A Proposal for Defining Limits & Continuity in Calculus

The goal is to introduce limits and continuity in an intuitive but formal way. Specifically, we aim to avoid (1) nested quantifiers, (2) auxiliary variables (ε-δ), and (3) inverse images. We aim to (1) chunk the definition into understandable parts, (2) use evocative language, and (3) create intermediate waypoints, the comprehension of which can be independently assessed.

Basic definitions

For a subset A of the real numbers, we denote by span A the smallest closed interval containing A. *We take the existence of spans as an axiom.*

We say that a sequence/continuum of nested intervals collapses to a point x if x is the unique point in all of the intervals.

Continuity

A function $f(x)$ is continuous at a if the sequence of intervals

$$\text{span}\, f(a-1/n,\ a+1/n)$$

collapses to $f(a)$.

(We can replace $(a-1/n,\ a+1/n)$ with any sequence of intervals collapsing to a.)

This can be modified to give definitions of the limit and left/right limit.

Limits

We define the n-th tail of the sequence x_i as the sub-sequence starting at x_n.

We say lim $x_i = x$ if the spans of the tails collapse to x.

Sample questions

Consider $f(x) = x^2$. For $n=1,2,3$, find the intervals $(3-1/n,\ 3+1/n)$. Determine the corresponding sets $f(3-1/n,\ 3+1/n)$ and the span $f(3-1/n,\ 3+1/n)$. Which real numbers are in all of the spanning sets span $f(3-1/n,\ 3+1/n)$? Is f continuous at 3? Extra credit: prove $f(x)$ is continuous at 3.

Let $x_i = (-1)^i/(i^2+1)$. Find the first few terms of the first three tails of (x_i), and the corresponding spans. Does the limit of (x_i) exist? What is the limit? Extra credit: prove it.