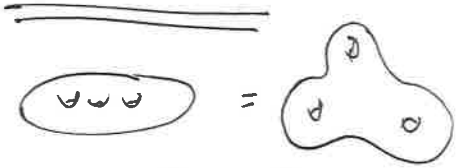
Math 9200 MCG

8/20/25

Book: primer

HW: fortnightly 60%

Final project 40%

S = 

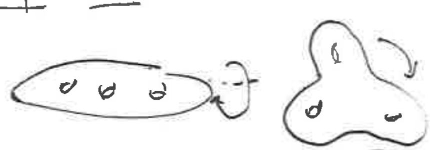
event horizon, free surface (fluid)

solns to polynomials $x^n + y^n = z^n$

$\mathrm{Mod}(S)$ = symmetries of S.

$= \pi_0 \,\mathrm{Homeo}^+(S)$.

Sample elts

① 

order 3 on  ?

② Dehn twists.

Examples

① $\mathrm{Mod}(S^2)$

② $\mathrm{Mod}(T^2)$

③ $\mathrm{Mod}(S_2)$ harder

$\mathrm{Mod}(S_3)$ very hard!

3 reasons

① GGT DNB

$\mathrm{Mod}^{\pm}(S_g) \cong \mathrm{Out}\,\pi_1(S_g)$

topology algebra

geometry

② Topology

eg. $I \to E$
$\downarrow$
S^1

{S-bundles over B}

$\longleftrightarrow \{\pi_1 B \to \mathrm{Mod}(S)\}$

e.g. $B = S^1$

Agol, Wise, Perelman, Thurston

Essentially: all 3-manifolds arise this way

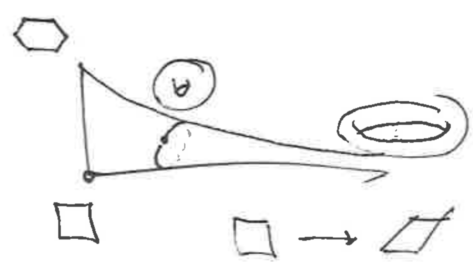
Donaldson: All symp. 4-mans arise this way

③ Alg. geom

mod. sp.

$\mathrm{Mod}(S_g) = \pi_1 M_g$

$g=1$ $y^2 = x(x-a)(x-b)$



④ Dynamics

$\sigma_1 \sigma_2^{-1}$

Connects to:

Number theory

Group thy

Rep thy

Graph thy

$\mathbb{C}$ - analy.

Alg. top.

Hyp. geom.

Group cohom

8/22/25

- Think of course as flipped.
- Learn how to think about the subject.
- Practice asking questions.

$\mathrm{Mod}(S) = \pi_0 \mathrm{Homeo}^+(S)$

curves : $\mathrm{Mod}(S)$ ::
vectors : $GL_n \mathbb{Z}$

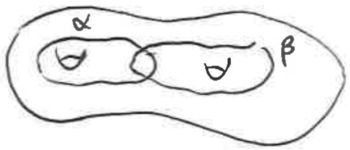
Chapter 1 Curves, surfaces, & hyp. geom.

Highlights

① Geom. int #

a, b = homotopy classes

$$i(a,b) = \min_{\alpha \in a,\, \beta \in b} |\alpha \cap \beta|$$



② Bigon crit.

min pos. $\Leftrightarrow$ no 

example

③ Change of coords

ex. $i(a,b) = 1 \Rightarrow$ std pic

$\rightsquigarrow$ braid rel.

① Geom int number

Obs 1. $i(a,b) \neq |\hat{i}(a,b)|$

use bigon crit!

Obs 2. $[a] = [a'] \not\Rightarrow i(a,b) = i(a',b)$

Torus case

Fact. On T^2

$$\left\{ \begin{matrix} \text{homot. class} \\ \text{scc} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{prim elts} \\ \text{of } \mathbb{Z}^2 \end{matrix} \right\} / \pm$$

$\rightarrow$ is H_1.

$\leftarrow$ is construction

Fact. $i((p,q),(r,s)) = \left| \begin{matrix} p & r \\ q & s \end{matrix} \right|$

Pf. Check for $(p,q) = (1,0)$

General case: apply A s.t. $A\binom{p}{q} = \binom{1}{0}$.

A: lin map of T^2.

Farey graph.

③ Change of coords

ex1 α, β nonsep $\Rightarrow \exists\, h \in \mathrm{Homeo}(S_g)$ s.t. $h(\alpha) = \beta$

pf: cut, classif. of surfs.

ex2. $\alpha \cap \beta = 1$

$\boxed{\mathbb{H}^2 \text{ geometry}}$

8/25/25

· Practice asking q's

Chapter 1 Curves, surfaces, and hyp. geom.

Highlights

① geom int #

② bigon crit.

③ change of coords.

The torus: change of coord

Fact. All (essent.) s.c.c. on T^2 are nonsep.

Fact. All (nonsep.) s.cc on T^2 are equiv. under Homeo.

Fact. $\{\text{s.c.c.}\}/\text{homot.} \longleftrightarrow \{\mathbb{Z}^2 \setminus 0\}/\text{mult.}$

$\longrightarrow H_1$

$\longleftarrow$ construction.

Fact. $i((p,q),(r,s)) = \left|\begin{matrix} p & r \\ q & s \end{matrix}\right|$

Pf. Check for $(p,q) = (1,0)$.
General case: $A\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Farey graph.

$\mathbb{H}^2$

$\frac{1}{y}\, d\mathrm{Euc}$

$\frac{1}{1-r^2}\, d\mathrm{Euc}.$

3 types of isometries.

ideal triangles.

If $\chi(S) < 0$ then S can be given a hyp. struct. and $\tilde{S} \cong \mathbb{H}^2$. (Cartan-Hadamard)

Cov. space theory

Fact. $1 \neq \alpha \in \pi_1(S_g)$ $g > 1$

$\Rightarrow C(\alpha) \cong \mathbb{Z} = \langle \alpha_0 \rangle$

α_0 = root of α.

Alg top. $\pi_1(S) \longrightarrow \mathrm{Homeo}(\tilde{S})$
$\longrightarrow \mathrm{Isom}^+(\mathbb{H}^2)$

Fact 1. $\alpha \to$ hyp/lox isom.

Fact 2. $C(\text{lox}) \cong \mathbb{R}$

Closed curves & geodesics

$\left\{\begin{matrix}\text{conj. classes} \\ \text{in } \pi_1\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{free homot. classes} \\ \text{of oriented curves}\end{matrix}\right\}$

$\longleftarrow$ pt pushing

$\longrightarrow$

$\left\{\begin{matrix}\text{elts of} \\ \text{conj. class } a\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{lifts to } \mathbb{H}^2 \\ \text{of } a\end{matrix}\right\}$

$\left\{\begin{matrix}\text{free hom.} \\ \text{classes of simple}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{(simple)} \\ \text{geodesics}\end{matrix}\right\}$

$\longrightarrow$ straight line homotopy

Bigon crit

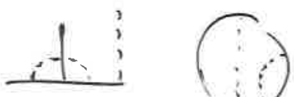
No bigons $\Rightarrow$ lifts int. ≤ 1 time

$\Rightarrow$ min pos.

8/27/25

- Practice asking q's
- OH Tu 9:30-10:30
  By appt.

Hyp. space



$\frac{1}{y}\, d_{Euc}$. $\quad \frac{1}{1-r^2}\, d_{Euc}$.

$\triangle$'s are thin, ideal $\triangle$

3 types of isometries$^+$

$\longleftrightarrow$ Möbius tr's.

Fact. $\chi(S) < 0 \implies$ S can be given a complex/hyp str.

Pf. Alg top + uniformization or Cartan-Hadamard.

$\mathrm{Isom}^+ \mathbb{H}^2 = SL_2\mathbb{R}/\pm$

$\implies$ can classify isometries by matrices.

Curves & univ cover

Fact. $1 \neq \alpha \in \pi_1(S)$ $\quad \chi(S) < 0$.

$\implies C(\alpha) \cong \mathbb{Z} = \langle \alpha_0 \rangle \quad \alpha_0^k = \alpha$

Alg top $\pi_1(S) \to \mathrm{Deck} \leq \mathrm{Isom}^+ \mathbb{H}^2$ or $\mathrm{Mob}\,\triangle$.

Fact 1. S compact $\implies \alpha \leftrightarrow$ lox $\quad$ no elliptic!

Fact 2. $C(\text{lox}) \cong \mathbb{R}$

Fact 3. $\pi_1/\text{conj} \leftrightarrow$ scc/free homot.

Fact 4. conj class $\leftrightarrow$ lifts to $\mathbb{H}^2$

π_1 acts on both in same way

both in corresp with $\pi_1/\langle\alpha_0\rangle$

Fact 5. scc/free homot $\longleftrightarrow$ simple geod.

$\to$ straight line homot.

Bigon criterion

Pf 1

Lemma. No bigons $\implies$ lifts int. ≤ 1

Latter $\implies$ min pos.

Pf 2.

8/29/25

- Practice asking q's.
- HW on web site

Curves & univ. cover

Fact 1. S compact.
$\alpha \in \pi_1(S)$ $\alpha \neq 1$
$C(\alpha) \cong \mathbb{Z}$

Fact 2. $\alpha \subseteq S$ s.c.c.

$\left\{ \begin{matrix} \text{elevations} \\ \text{of } \alpha \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{cosets} \\ \text{of } \alpha \end{matrix} \right\}$

$\longleftrightarrow \left\{ \begin{matrix} \text{conj} \\ \text{of } \alpha \end{matrix} \right\}$

Fact 3. $\alpha \subseteq S$ c.c.
is homot. to unique geod.

Bigon crit

Prop. $\alpha, \beta \subseteq S$ s.c.c. . .

Cor. Geodesics are in min pos.

Lemma. No bigons $\Longleftrightarrow$ elevations int. at most once.

Pf of BC 1. Apply lemma.

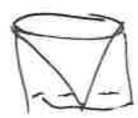
Pf of BC 2.

Chap 2 MCG basics

$Mod(S) = Homeo^+(S, \partial S) / \text{homot.}$

Alex. Lemma. $Mod(D^2) = 1$

$F(x,t) = \begin{cases} (1-t)\varphi(x/1-t) & 0 \le |x| \le 1-t \\ x \end{cases}$



also works for punctured disk!

Prop. $Mod(S_{0,3}) \cong \Sigma_3$

Lemma. α, α' simple arcs connecting same end pts then $\alpha \sim \alpha'$.

Pf of Prop. Isotope, cut.

$\boxed{\text{Annulus}}$ $\tilde{\varphi}(\tau \cdot x) = \varphi_*(\tau)\tilde{\varphi}(x)$

$\Rightarrow \tilde{\varphi}(\tau \cdot x) = \tau \cdot \tilde{\varphi}(x)$.

9/1/25

- Practice q's.
- OH Tue 9:30
- HW 1 9/12.

Chap 2 MCG Basics

$\mathrm{Mod}(S) = \pi_0 \mathrm{Homeo}^+(S, \partial S)$
$= \mathrm{Homeo}^+ / \text{isot.}$
$= \mathrm{Homeo}^+ / \mathrm{Homeo}_0$

or Diff...

Alex. Lemma $\mathrm{Mod}(D^2) = 1$

Pf. $F(x,t) = \begin{cases} t\,\varphi(x/t) & |x| \le t \\ x & \text{o.w.} \end{cases}$

Lemma. $\mathrm{Mod}(S_{0,1}') = 1$

same pf.

Prop. $\mathrm{Mod}(S_{0,3}) \xrightarrow{\cong} \Sigma_3$

Lemma. Two simple arcs in $S_{0,3}$ w/ same endpts are isotopic.

Pf. Use pf of bigon crit to make disjoint, cut

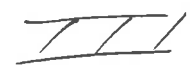
Pf of Prop. Surj hom ✓

Inj: f pure $\Rightarrow f(a) = a$.

$\rightsquigarrow$ cut ...

Prop. $\mathrm{Mod}(A) \cong \mathbb{Z}$.

Pf. $\mathrm{Mod}(A) \to \mathbb{Z}$

given by 

surj by Dehn twist.

inj: same. or straight line.

Prop. $\mathrm{Mod}(T^2) \xrightarrow{\cong} SL_2\mathbb{Z}$

Pf. Surj: Dehn twists.

Inj: $K(G,1)$

Prop. $\mathrm{Mod}(S_{0,4}) \cong PSL_2\mathbb{Z} \ltimes (\mathbb{Z}/2 \times \mathbb{Z}/2)$

$\cong PSL_2\mathbb{Z}[2] \times A_4$.

9/3/25

- Practice asking q.
- OH Tu & appt.
- HW1

Ch2. MCG Basics

$\mathrm{Mod}(D^2) = \mathrm{Mod}(S_{0,1}) = 1$.

$\mathrm{Mod}(S_{0,2}) \cong \mathbb{Z}/2$

$\mathrm{Mod}(S_{0,3}) \cong \mathbb{Z}/3$

$|\mathrm{Mod}(S_{0,4})| = \infty$.

∞ many curves.

Prop. $\mathrm{Mod}(A) \cong \mathbb{Z}$.

Pf. surj

inj.

Thm. $\mathrm{Mod}(T^2) \xrightarrow{\cong} SL_2\mathbb{Z}$

surj #1 $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

surj #2 $\begin{pmatrix} p & r \\ q & s \end{pmatrix}$ c.o.c.

inj. $K(G,1)$

Alex method.

Want: $f \in \mathrm{Mod}(S)$ det by action on (finitely many) curves.

Attempt #1

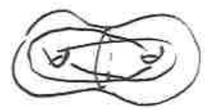
If $\{c_1, \dots, c_k\}$ fill S (cut into disks)

and $f(c_i) = c_i$ then $f = \mathrm{id}$.

 X

Attempt #2

Also pres. orientation.

 X

Thm. $S = S^b_{g,n}$

~~[illegible]~~ $\varphi \in \mathrm{Homeo}^+(S)$

$\{\gamma_i\}$ curves/arcs

min pos

pairwise not $\sim$

no 3 int. eachother.

$\varphi(\gamma_i) \sim \gamma_j \quad \forall i$.

Then $\varphi \sim \varphi'$ s.t. $\varphi'(\gamma_i) = \gamma_{j(i)}$

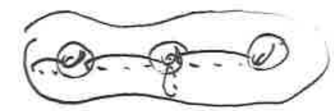
$\Rightarrow \varphi' \rightsquigarrow$ aut of $\cup \gamma_i$ (graph)

φ_*

If φ' fixes $\cup \gamma_i$ ~~[illegible]~~ as an oriented graph

then $\varphi \sim \mathrm{id}$.

otherwise $[\varphi]$ has finite order.

e.g. 

$\sigma = \mathrm{id}$

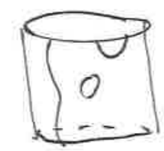
$\Rightarrow \varphi_* = \mathrm{id}$.

$\Rightarrow \varphi \sim \mathrm{id}$.

Idea. Move curves one by one.

γ_1 no prob.

γ_2



9/5/25

- Practice q's
- HW 1

Alex. Method

$S = S_{g,n}$

$\varphi \in \mathrm{Homeo}^+(S, \partial S)$

$\gamma_1, \dots, \gamma_n$ scc/arcs

① min pos
② non-homot.
③ no 3 all int.

If $\varphi(\gamma_i) \sim \gamma_i \ \forall i$

then $\varphi(\cup \gamma_i) \overset{\text{amb}}{\sim} \cup \gamma_i$

$\rightsquigarrow \varphi_*$ = aut of dir. (ribbon) graph $\cup \gamma_i$

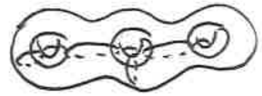
$\rightsquigarrow$ If γ_i fill

$\Rightarrow |[\varphi]| = |\varphi_*|$.

Idea. Induct on n.

$n=1$ ✓ iso. ext.

$n=2$ homotope $\varphi\gamma_2 \to \gamma_2$ without messing up γ_1

example. 

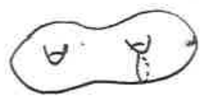
If $\varphi(\gamma_i) \sim \gamma_i \ \forall i$

then $\varphi \sim id$.

Alex method $\rightsquigarrow \varphi_*$

Want $\varphi_* = id$.

Chap 3: Dehn twists



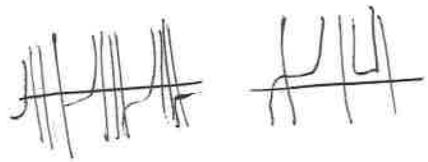
How to draw picts:

To find $T_a(b)$

draw $i(a,b)$ copies of a...

Prop. $i(T_a^k(b), b) = |k|\, i(a,b)^2$

Cor. Dehn twists have ∞ order

Pf. 

Facts about Dehn twists

① $T_a = T_b \Leftrightarrow a = b$

② $f T_a f^{-1} = T_{f(a)}$

③ $f \leftrightarrow T_a \Leftrightarrow f(a) = a$

④ a, b nonsep $\Rightarrow T_a$ conj T_b

⑤ $T_a \leftrightarrow T_b \Leftrightarrow i(a,b) = 0. \Leftrightarrow T_a(b) = b.$

Thm. $g \geq 3 \quad Z(\mathrm{Mod}(S_g)) = 1$.

Pf. Above pic + Fact ③

· Practice q's
· OH Tue, appt, stop by
· HW 1 Fri

Chap 3. Dehn twists

Fact 1. $T_a = T_b \Leftrightarrow a = b$

Fact 2. $f T_a f^{-1} = T_{f(a)}$

Fact 3. $f \leftrightarrow T_a \Leftrightarrow f(a) = a$.

Fact 4. a, b nonsep $\Rightarrow T_a \sim T_b$

Fact 5. $i(a,b) = 0 \Leftrightarrow T_a(b) = b$
$\Leftrightarrow T_a \leftrightarrow T_b$

Braid relation

Prop. $i(a,b) = 1 \Rightarrow$
$T_a T_b T_a = T_b T_a T_b$

Prop. Converse! $a \neq b$
$T_a T_b(a) = b$.
$i(a,b) =$

Groups gen by 2 DT's

Prop. $i(a,b) \geq 2 \Rightarrow \langle T_a, T_b \rangle \cong F_2$.

Ping pong lemma. $G \circlearrowright X$ set.
$a, b \in G$
$X_a, X_b \neq \emptyset$ disj.
$a^k(X_b) \subseteq X_a$ vice versa.
Then $\langle a, b \rangle \cong F_2$.

$SL_2\mathbb{Z}$

Pf of Prop.
$X_a = \{c : i(c,b) > i(c,a)\}$
X_b similar.
Say $i(a,c) > i(a,b)$

$$\begin{aligned} i(T_a^k(c), b) &\geq |k|\, i(a,b)\, i(a,c) - i(b,c) \\ &\geq 2|k|\, i(a,c) - i(b,c) \\ &\geq 2|k|\, i(a,c) - i(a,c) \\ &= (2|k|-1)\, i(a,c) \\ &\geq i(a,c) \\ &= i(a, T_a^k(c)) \end{aligned}$$

9/8/25

$n=1$

Prop. $A = \{a_1, \ldots, a_n\}$ $i(a_i, a_j) = 0$.
$T_A = \prod T_{a_i}^{e_i}$ e_i all $+, -$

$$\left| i(T_A(b), c) - \sum |e_i|\, i(a_i, b)\, i(a_i, c) \right| \leq i(b,c)$$

One rep

Pf. $T_A(b) \cup b$ intersects
c in $\sum |e_i|\, i(a_i,b)\, i(a_i,c)$

$\sum \leq |(\beta \cup \beta') \cap \gamma| = i(T_A(b), c) + i(b,c)$

$\sum$ not too big.
$i(T_A(b), c)$ not too big: pic.

Cut / Cap / Include

Include $S \hookrightarrow S'$ closed $\rightsquigarrow \mathrm{Mod}(S) \to \mathrm{Mod}(S')$
usually inj.

Cap. $1 \to T_\partial \to \mathrm{Mod}(S) \to \mathrm{Mod}(\hat{S}) \to 1$.

Cut. $1 \to \langle T_{a_i} \rangle \to \mathrm{Mod}(S) \to \mathrm{Mod}(S - \cup a_i) \to 1$

9/10/25

- Practice q's
- HW1 Fri
- OH Tue, appt, etc.

Chapter 4 Generating MCG

$\mathrm{PMod}(S_{g,n}^b) = \ker\left(\mathrm{Mod}(S_{g,n}^b) \to \Sigma_n\right)$

Thm. $\mathrm{PMod}(S_{g,n}^b)$ is fg by Dehn twists.

$g \geq 1$: nonsep.

Humphries: 

$2g+1$ min. gen set. using twists.

Ingredient #1 Curve graph

$C(S)$ vertices
edges

Thm. $3g+n \geq 5 \Rightarrow C(S)$ conn.

i.e. S not $S_{1,n}$ $n \leq 1$
$S_{0,n}$ $n \leq 4$

Pf. Induct on $i(a,b)$.

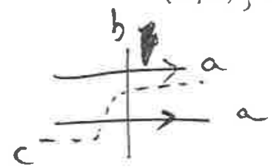
$i(a,b) = 0$ ✓

$i(a,b) = 1$ ✓ if $S \neq S_{1,n}$ $n<2$.

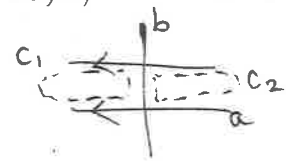
Assume $i(a,b) \geq 2$

Will find c with

$i(a,c), i(b,c) < i(a,b)$.



always ess.



Some c_i ess. unless $S = S_{0,4}$.

Cor. $N(S)$, $\hat{N}(S)$ connected.

Ingredient #2 Birman exact seq.

Forget: $\mathrm{Mod}(S,x) \twoheadrightarrow \mathrm{Mod}(S)$

$f \in \ker \rightsquigarrow \alpha \in \pi_1(S,x)$

Thm. $\chi(S) < 0$.

$1 \xrightarrow{P} \pi_1(S,x) \xrightarrow{F} \mathrm{Mod}(S,x) \to \mathrm{Mod}(S) \to 1$

P is: isotopy extension.

Fact. α simple $\Rightarrow P(\alpha) = T_a T_b^{-1}$

Fact. $h \in \mathrm{Mod}(S,x)$

$\mathrm{Push}(h(\alpha)) = h\,\mathrm{Push}(\alpha)\,h^{-1}$

Pf. Fiber bundle

$\mathrm{Homeo}^+(S,x) \to \mathrm{Homeo}^+(S) \xrightarrow{\varepsilon} S$

Thm. $\mathrm{PMod}(S_{0,n})$ f.g. by DT's.

- Practice q's
- HW 2
- OH Tue, etc.

9/12/25

Chap 4. Generating MCG.

Ingredient #0.

$G \circlearrowright X$ = conn. graph.

X/G one vertex, one edge ~ acts trans on orient. edges.

Choose $\overset{v}{\bullet}\!-\!\overset{w}{\bullet}$ & $t \in G$

s.t. $g(v) = w$.

Then $G = \langle \mathrm{Stab}\, v, t \rangle$

Pf. Say $|g| = d(v, g(v))$.

Induct on $|g|$.

Base $|g| = 0 \Rightarrow g \in \mathrm{Stab}\, v$ ✓

* Assume $|g| > 0$.

Choose path v — $h(v)$ — $g(v)$

$h \in \langle \mathrm{Stab}\, v, t \rangle$

* Base case $|g| = 1$ (edges v — $g(v)$, v — w)

Choose $r \in \mathrm{Stab}\, v$ s.t. $rg(v) = w$ $t^{-1}rg \in \mathrm{Stab}$.

Check hypotheses →

Ingredient #1

$\hat{N}(S)$ connected $g > 0$.

Ingredient #2: BES.

Thm. $\chi(S) < 0$

$$1 \to \pi_1(S,x) \to \mathrm{Mod}(S,x) \to \mathrm{Mod}(S) \to 1$$

Pf. $H_0 \to \mathrm{Homeo}(S)$

$\downarrow$

S

fiber bundle. $\rightsquigarrow$ LES.

Fact. $\mathrm{Push}(\alpha) = T_a T_b^{-1}$.

Pf of Thm

Base case $g = 1$. ✓.

Double induction.

If true for S_g, true for $S_{g,n}$. by BES.

Assume $g \geq 2$.

$\mathrm{Mod}(S_{g,n}) \circlearrowright \hat{N}(S_{g,n})$

Lemma ⇒

$\mathrm{Mod}(S_{g,n}) = \langle \mathrm{Stab}\, v, t \rangle$

- $t = T_a T_b$

$$\mathbb{Z} \to \mathrm{Stab}\ v \twoheadrightarrow \mathrm{Mod}(S_{g-1,n+2})$$

$\mathbb{Z} = T_v$

nonsep. ⇠ nonsep

9/15/25

- Practice asking q's.
- HW 2
- OH Tue etc.

Chap 4. Generating MCG.

Thm $PMod(S_{g,n})$ f.g. by Dehn twists.

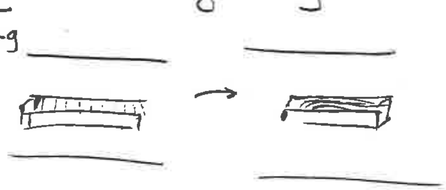
Thm (Lickorish) Every M^3 obtained from S^3 by Dehn surgery

Ing#1. Heegaard splitting.

Ing#2. Generation by DT

Ing#3. S^3 has (!) H.S. of genus g.

Σ_g



Pf of gen by twists

Ing#0. $G \circlearrowright X$

Need ① X connected.
② Understand stabs G_v.
③ Local moves

Thm. BES $\chi(S) < 0$.

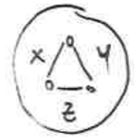
$$1 \to \pi_1(S) \to Mod(S,x) \to Mod(S) \to 1$$

Pf. Fiber bundle.

$$\begin{array}{ccc} Homeo(S,x) & \to & Homeo(S) \\ & & \downarrow \\ & & S \end{array}$$

Chap 5. Presentations...

Lantern relation $T_x T_y T_z = \prod T_{\partial_i}$



Pf #1: Alexander

Pf #2: Push

Thm. $Mod(S_{g,n}^b)$ is perfect when $g \geq 3$.

Pf. Ing #1. Gen. by DT about nonseps.
Ing#2. All such are conj.
Ing#3. Nonsep lanterns.

- Practice q's
- HW 2
- OH Tue etc.

9/17/25

Chap 5. Presentations...

Lantern relation

$T_x T_y T_z = \mathbb{1}$

Pf #1 Alex.

Pf #2 Push.

Thm. $g \geq 3$ $PMod(S_{g,n}^b)^{ab} = 1$

Ing #1. Dehn-Lickorish.

Ing #2. Lantern.

Ing #3. Nonsep lantern.

Thm. $Mod(S_{g,n}^b)$ fin. pres.

Ing #0. $G \circlearrowright X$ = ~~simply conn.~~ contract.

w.o. rotations.

- finitely many orbits of vertices, edges, faces.
- All G_v fin. pres. & G_e fin gen.

$\Rightarrow G$ fin pres.

Pf. Borel construction.

$U = K(G,1)$

$G \circlearrowright \tilde{U} \times X$

$(\tilde{U} \times X)/G \to X/G = K(G,1)$

fiber over v is $K(G_v,1) \times$ ~~v~~ v

or: $(\tilde{U} \times$ ~~v~~ $v) / G_v$

fiber over e is $K(G_e,1) \times e$

Hypotheses $\Rightarrow$ finite 2-skel

$\Rightarrow G$ fin pres.

$\blacksquare$

Ing #1. $A(S_{g,n}^b)$ ~~Connected~~ contractible.

Pf. Hatcher flow.

Pf of Thm.

Combine ingredients + induct.

Low-dim homology

- $H_1(PMod(S_{g,n}^b)) = 1$ $g \geq 3$
- Harer: $g \geq 4$

$H_2(Mod(S_g)) = \mathbb{Z}$

$H_2(Mod(S_g^1)) = \mathbb{Z}$

$H_2(Mod(S_{g,1})) = \mathbb{Z}^2$

$H_2(Mod(S_{g,n}^b)) = \mathbb{Z}^{n+1}$

Euler class

$e(E) \in H^2(B)$

S^1 or $\mathbb{R}^2$ $\to E$
$\downarrow$
B

- Sections
- $1 \to \mathbb{Z} \to E \to G \to 1$

central.

choose set theoretic section $\varphi: G \times \mathbb{Z} \to E$

$\leadsto f: G \times G \to \mathbb{Z}$.

$\varphi(g_1,n_1)\varphi(g_2,n_2) = \varphi(g_1g_2, n_1+n_2+f(n_1,n_2))$

- Practice q's
- HW 2
- OH Tue etc.

9/19/25

Chap 5. Presentations...

Today $H^*(\mathrm{Mod}(S))$.

Fact

$\left\{ \begin{matrix} S\text{-bundles} \\ \text{over } B \end{matrix} \right\}/\sim \;\longleftrightarrow\; \{\pi_1(B) \to \mathrm{Mod}(S)\}/\sim$

monodromy

eg. $B = S^1$

$B = S_h$.

A char. class of S-bundles is $\left\{ \begin{matrix} S\text{-bundles} \\ \text{over } B \end{matrix} \right\} \longrightarrow H^*(B)$

natural under pullback.

Fact $\left\{ \begin{matrix} \text{char classes} \\ \text{for } S\text{-bundles} \end{matrix} \right\} \longleftrightarrow H^*(\mathrm{Mod}(S))$.

$\longleftarrow$ monodromy.

We know:

$H^1(\mathrm{Mod}(S_{g,n}^b)) = 0 \quad g \geq 3$

$\Rightarrow$ no 1-dim char classes.

Harer $H^2(\mathrm{Mod}(S_{g,n}^b)) = \mathbb{Z}^{n+1}$

So $H^2(\mathrm{Mod}(S_g)) = \mathbb{Z}$

$H^2(\mathrm{Mod}(S_{g,1})) = \mathbb{Z}^2$

An elt of H^1 determined by value on bundles over S^1

For H^2: bundles over S_h.

Thm. The generator of $H^2(\mathrm{Mod}(S_g))$ is signature.

Euler class.

For S^1-bundle, $e \in H^2(B)$ measures twistedness

$e(\text{trivial}) = 0$. $e(TS_g) = \chi(S_g)$.

Given

$1 \to \mathbb{Z} \to G \to Q \to 1$ ($\mathbb{Z}$ central.)

$e \in H^2(Q)$ measures twistiness.

$e \neq 0 \Leftrightarrow$ no section.

The seq.

$\boxed{\text{no section by torsion}}$

$1 \to \mathbb{Z} \to \widetilde{\mathrm{Homeo}^+(S^1)} \to \mathrm{Homeo}^+(S^1) \to 1$

$\rightsquigarrow e \in H^2(\mathrm{Homeo}^+(S^1))$

euler class.

Lifting gives

$\mathrm{Mod}(S_{g,1}) \longrightarrow \mathrm{Homeo}^+(S^1)$

$\rightsquigarrow e \in H^2(\mathrm{Mod}(S_{g,1}))$

this is the other generator.

9/22/25

- Practice q's
- OH Tue ppd.
- HW 2 Fri.

Chap 6. Symplectic rep.

$g \geq 1$

$x_1, y_1, \dots, x_g, y_g$ basis $\mathbb{R}^{2g}$

$\omega = \sum dx_i \wedge dy_i$

std symplectic form.

unique nondeg, alt, bilin. form

$Sp_{2g}(\mathbb{R}) \subseteq GL_{2g}(\mathbb{R})$ $\quad A^*\omega = \omega$

$$SE_{ij} = \begin{cases} I_{2g} + e_{ij} & i = \sigma(j) \\ I_{2g} + e_{ij} - (-1)^{i+j} e_{\sigma(j)\sigma(i)} \end{cases}$$

Thm. $Sp_{2g}(\mathbb{Z}) = \langle SE_{ij} \rangle$

Example. $H_1(S_g; \mathbb{R})$, $\hat{i}$

$\rightsquigarrow$ geom. sympl. basis.

Prop. $v \in H_1(S_g; \mathbb{Z})$ rep by scc

$\iff v$ primitive.

Euc. alg.

Sympl. rep

$\psi : Mod(S_g) \to Sp_{2g}\mathbb{Z}$

Koberda-Pengitore: $Mod(S_g)$ not linear

Prop. $\psi(T_b^k[a]) = [a] + k\,\hat{i}(a,b)[b]$

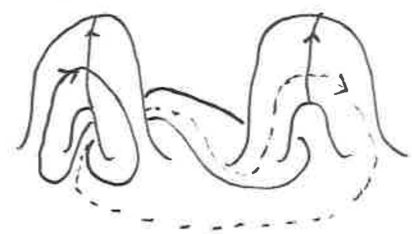
Thm. ψ is surjective

Pf #1. Sp gen by transvections

$T_a \longmapsto \tau_{[a]}$

Pf #2. SE_{ij} are transvections

& factor mixes:



Pf #3. Given a sympl. basis, realize it geometrically

Congruence subgps

Thm $Mod(S_g)[m]$ torsion free for $m \geq 3$.

- Practice q's
- HW 2 Fri
- OH Thu 9:30 etc

9/24/25

Thm. $\psi : \mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z}$ surj.

Torelli = hard part.

Pf idea: Given symp. basis x_i, y_i

Make a geometric symp. basis a_i, b_i $i(c,d) = |\hat{i}(c,d)|$ $[a_i], [b_i] = x_i, y_i$.

Base step: x_1

Prop. $v \in H_1(S_g; \mathbb{Z})$, $v \neq 0$.

v rep by scc $\Leftrightarrow$ v prim.

Pf. $\Rightarrow$ c.o.c.

$\Leftarrow$ Say $(v_1, w_1, \ldots, v_g, w_g)$ prim.

Each (v_i, w_i) rep. by $\gcd(|v_i|, |w_i|)$ $\parallel$ curves on i^{th} handle.

Surger, Euclid alg, induct on g.

Next step: y_1

Induct on g.

Torsion-free subgps

Serre: $SL_n\mathbb{Z}[m]$ torsion free $m \geq 3$.

Thm. $\mathrm{Mod}(S_g)[m]$ torsion-free $m \geq 3$.

Pf. $1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g)[m] \to Sp_{2g}(\mathbb{Z})[m] \to 1$

Residual finiteness.

G resid. finite:

① $\cap H = 1$

② $\forall g \in G$ $\exists$ $G \to F \ldots$

Thm. $\mathrm{Mod}(S)$ is resid. finite.

Pf. Case 1. $\psi(f) \neq I$

Case 2. $|f| = \infty$.

Choose hyp. metric

$\rightsquigarrow \rho : \pi_1(S) \to PSL_2\mathbb{R}$

$\to PSL_2 A$

subrings of $\mathbb{R}$ resid. finite.

$\to PSL_2 F$.

length of a curve is fn of trace $|\mathrm{tr}|(\gamma) = 2\cosh(\ell(\gamma)/2)$

$|f| = \infty \Rightarrow f$ changes length of one curve c.

Choose F to detect this.

$H_0 = \ker$.

H = char subgp of fin. index.

$\mathrm{Mod}(S_g) \to \mathrm{Out}\,\pi_1(S_g)$

$\to \mathrm{Out}(\pi_1(S_g)/H)$.

9/26/25

- Practice asking q's.
- HW 3
- OH Tue etc.

Chap 6. Symplectic...

G is resid. finite if

① $\cap H = 1$

$\Leftrightarrow$ ② $\forall g \; \exists \; G \to F \ldots$ f.g. linear

Thm. $SL_n\mathbb{Z}$ resid. fin. Malcev

Thm. $\mathrm{Mod}(S)$ resid. finite.

Pf. Case 1. $\psi(f) \neq 1 \rightsquigarrow$ Mal'cev.

Case 2. $\psi(f) = 1 \Rightarrow |f| = \infty$.

Ing 1 Choose deck gp $\rho: \pi_1(S) \to PSL_2\mathbb{R}$

$\to PSL_2 A$

Mal'cev $\Rightarrow$ A res. fin

$\rightsquigarrow \{\rho_F : \pi_1(S) \to PSL_2 F\}$

Ing2. $|f| = \infty \Rightarrow f$ changes length of some ess scc.

Ing3. $\ell(c) \leftrightarrow \mathrm{tr}\, \rho(c)$ $\quad |\mathrm{tr}|(\gamma) = 2\cosh(\ell(\gamma)/2)$

Given f find ρ_F. $\rightsquigarrow \ker H_0 \rightsquigarrow$ char subgp of finite index

$$\mathrm{Mod}(S) \to \mathrm{Out}\, \pi_1(S) \to \mathrm{Out}\, \pi_1(S/H)$$

Torelli gp

$$1 \to \mathcal{I}(S_g) \to \mathrm{Mod}(S_g) \to Sp_{2g}\mathbb{Z} \to 1$$

All $\mathbb{Z}HS^3$ arise from $\mathcal{I}(S_g)$!

Morita.

Fact. $\mathcal{I}(S_g)$ torsion free.

Types of elts

① sep
② bp
③ fake bp
csip.
④ pushes.

Prop. $\mathcal{I}(S_g)$ acts trans. on $\{c : [c] = x\}$.

Pf. sep / nonsep.

- Practice asking/answering q's.
- HW 3
- OH Tue etc.

Chap 6. Torelli group.

Generators $\boxed{\text{Heegaard splittings}}$

Birman '71 gens for $\mathcal{I}(S_g)$

Powell '78 seps & BP's.

Johnson '79 BP's.

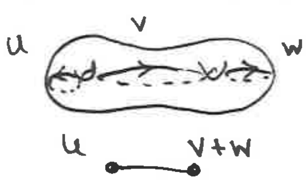
Johnson '80 seps don't.

Putman '07 GGT pf.

Hatcher-M '12 $C_x(S_g)$ conn.

vertices: scc rep'ing x.
edges: disj

BBM '07 $B_x(S_g)$



Thm BBM '07

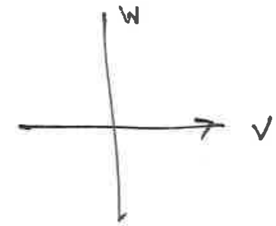
$B_x(S_g)$ contractible.

Cells of B_x: oriented multicurves s.t.

① every curve is in a nonneg rep. of x.

② only way to get 0 is 0.

Pf.



Drain Surg$(tv+(1-t)w)$

Prop. $\mathcal{I}(S_g)$ acts trans on vertices of $C_x(S_g)$.

Pf. Geom. sympl. basis.

1983 Johnson $\mathcal{I}(S_g)$ f.g.

Gives 2^g BP maps.

Shows they generate a normal subgp of MCG.

Putman '11 $O(g^3)$ gens.

Q. Is $\mathcal{I}(S_g)$ f.p.

Minahan-Putman '25
H_2 f.g.

9/29/25

Johnson Homom

$\tau : \mathcal{I}(S_g^1) \to A$

$\ker \tau = \mathcal{K}(S_g^1)$.

$H = H_1(S_g^1; \mathbb{Z})$

$A = \Lambda^3 H$

e.g. $(x_1+x_2)\wedge y_1 \wedge y_2$
$= x_1\wedge y_1\wedge y_2 + x_2 \wedge y_1 \wedge y_2$
$= -y_1 \wedge x_1 \wedge y_2 + x_2\wedge y_1 \wedge y_2$

$\tau(T_c T_d^{-1}) = (\sum a_i \wedge b_i)\wedge c$

3 constructions.

① Algebra

$\Gamma = \pi_1(S_g^1)$

$$1 \to \underbrace{\Gamma'/\Gamma''}_{=\Lambda^2 H} \to \Gamma/\Gamma'' \to \underbrace{\Gamma/\Gamma'}_{=H} \to 1$$

$f \rightsquigarrow \tau(f) \in \mathrm{Hom}(H, \Lambda^2 H)$

$f(\tilde{x})(\tilde{x})^{-1} \in \Lambda^2 H$.

10/1/25

- Practice asking/answering q's.
- HW 3
- OH Tue etc.

Chap 6 Torelli.

Johnson homom.

$\tau : \mathcal{I}(S_g^1) \longrightarrow \Lambda^3 H$

Two facts: ① τ surj

② $K \subseteq \ker \tau$

Construction #1

$\Gamma = \pi_1(S_g^1)$

$\Gamma^{(i)} = [\Gamma, \Gamma^{(i-1)}]$

$$1 \to \Gamma^{(1)}/\Gamma^{(2)} \to \Gamma/\Gamma^{(2)} \to \Gamma/\Gamma^{(1)} \to 1$$

Fact. $\Gamma^{(1)}/\Gamma^{(2)} \xrightarrow{\cong} \Lambda^2 H$

$[a,b] \longmapsto [a] \wedge [b]$

Define $\tau(f)$: $H \longrightarrow \Lambda^2 H$

$x \longmapsto f(\tilde{x})\tilde{x}^{-1}$

Fact. $\Lambda^3 H \subseteq \mathrm{Hom}(H, \Lambda^2 H) \cong H \otimes \Lambda^2 H$

$a \wedge b \wedge c \longmapsto a \otimes (b \wedge c) + b \otimes (c \wedge a) + c \otimes (a \wedge b)$.

Prop. $K \subseteq \ker \tau$

Pf. $T_c(x) = \gamma x \gamma^{-1}$

$T_c(x)x^{-1} = [\gamma, x]$

$\in \Gamma^{(2)}$

Prop. $\tau(BP) \neq 0$.

Pf. $f = T_d T_e^{-1}$

$f(\alpha_i) = \delta \alpha_i \delta^{-1}$
$\beta_i \beta_i$

$f(\alpha_{k+1}) = \underbrace{\delta \varepsilon^{-1}}_{\prod_{i=1}^{k}[\alpha_i, \beta_i]} \alpha_{k+1}$

$f(\alpha_1)\alpha_1^{-1} = [\delta, \alpha_1] \longleftrightarrow [\beta_{k+1}] \wedge [\alpha_1]$
$\phantom{f(\alpha_1)\alpha_1^{-1} = [\delta, \alpha_1] \longleftrightarrow [} y_{k+1} x_1$

$\rightsquigarrow \beta_1 \wedge \beta_{k+1} \wedge \alpha_1$

$= \alpha_1 \wedge \beta_1 \wedge [d]$

Construction #2

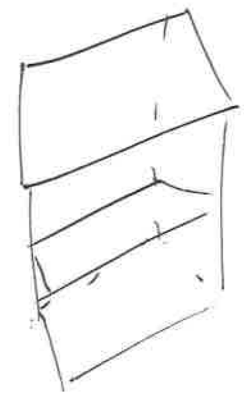
$f \rightsquigarrow M_f \rightarrow T^{2g}$

$[M_f] \longmapsto H_3 T^{2g} = \Lambda^3 H$.

Construction #3

Triple Intersection form in M_f.

$H_3 \longrightarrow H_2(M_f) = H^1(M_f)$



Abelianization

$\binom{2g}{2} + \binom{2g}{1} + \binom{2g}{0}$

$H_1(\mathcal{I}(S_g^1)) \cong \Lambda^3 H \times (\mathbb{Z}/2)$ *

$f \in \mathcal{I} \longrightarrow \mathbb{Z}HS^3 \longrightarrow$ Rochlin invt $\in \mathbb{Z}/2$.

$M^3 = \partial X \quad \sigma(X)/8 \mod 2$

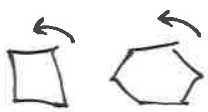
10/3/25

- Practice ask/answer
- HW 4
- OH Tue etc.

Chap 7. Torsion

<u>Thm</u>. $f \in \mathrm{Mod}(S)$ $|f| < \infty$
$\Rightarrow f$ rep. by φ, $|\varphi| = |f|$. (φ = isom.)

<u>Idea</u>. $\langle f \rangle \circlearrowright \mathrm{Teich}(S_g) \cong \mathbb{R}^N$
$\Rightarrow$ fixed pt $\rightsquigarrow$ hyp metric

Torus case: 

Sphere case: 

<u>Thm</u>. $f \in \mathrm{Mod}(S_g)$, $|f| < \infty$.
$\Rightarrow \Psi(f) \neq \mathrm{Id}$.

<u>Pf</u>. $g=1$ above.
$g=2$. iso fixed pts.

$$L(\varphi) = \#\text{ fixed pts} \geq 0$$
$$= \sum_{i=0}^{2} (-1)^i \mathrm{Tr}(H_i \to H_i)$$
$$= 1 - 2g + 1 = 2 - 2g < 0.$$

<u>Thm</u> (Hurwitz '93) $X \cong S_g$
$$|\mathrm{Isom}^+(X)| \leq 84(g-1).$$
realized for ∞ many g $\quad g=3$. $PSL_2(\mathbb{Z}/7)$

<u>Thm</u>. $\varphi \in \mathrm{Isom}^+(X)$
$\Rightarrow |\varphi| \leq 4g+2$.

<u>Prop</u>. Isom X finite.

<u>Pf</u>. compact top. gp (Arzela-Ascoli)
WTS discrete. ie.
$\mathrm{Isom}(X) \cap \mathrm{Homeo}_0(X) = 1$
$\varphi \in \cap \Rightarrow \tilde{\varphi}$ bdd dist. from id.

<u>Orbifolds</u>

$Y = X/G$ $\quad G$ = finite subgp of Homeo.
$Y = (S, P)$

$$\chi(Y) = \chi(S) - |P| + \sum \frac{1}{p_i}$$
$$= \chi(S) + \sum\left(\frac{1}{p_i} - 1\right).$$

$$\chi(X) = |G|\,\chi(Y).$$

$$\mathrm{Area}(Y) = -2\pi\,\chi(Y).$$

<u>Thm</u>
Smallest hyp orb has
$\chi = -1/42$ $\quad (2,3,7)$

<u>Pf of $84(g-1)$</u>
$$\chi(X/G) \leq -1/42.$$
$$(2-2g)/|G| \leq -1/42$$
$$\Rightarrow |G| \leq 84(g-1).$$

<u>Pf of $4g+2$</u> $\quad g(Y) = 0$.

$(2, 2g+1, 4g+2)$
LCM of any $|P|-1$ is $|G|$.

<u>Thm</u>. Every finite gp is a subgp of some MCG.

Generating w/ torsion (Luo).

10/6/25

- Practice asking/answering
- HW 4 Monday
- OH Tue etc.

Chap 7. Torsion.

Nielsen real:

finite subgps of Mod(S) $\longleftrightarrow$ finite gps of isometries

both directions nontrivial!

Thm. X = hyp surf.
$\mathrm{Isom}(X)$ finite.

Pf. compact top gp (Arz-Asc.)
WTS discrete i.e.
$\mathrm{Isom}(X) \cap \mathrm{Homeo}_0(X) = 1$
$\varphi \in \cap \Rightarrow \tilde{\varphi}$ bdd dist fom id
$\Rightarrow \varphi = 1$

Thm (Hurwitz)
$|\mathrm{Isom}^+(X)| \leq 84(g-1)$

Thm. $\varphi \in \mathrm{Isom}^+(X)$
$\Rightarrow |\varphi| \leq 4g+2$.

Orbifolds

$Y = X/G \quad G \leq \mathrm{Isom}^+(X)$ finite.

$Y = (S, P)$

$\chi(Y) = \chi(S) - |P| + \sum \frac{1}{p_i}$
$= \chi(S) + \sum\left(\frac{1}{p_i} - 1\right)$

$\chi(X) = |G|\,\chi(Y)$

$\mathrm{Area}(Y) = -2\pi\,\chi(Y)$

$\Rightarrow$ Gauss-Bonnet

Thm Smallest hyp. orb.
has $\chi = -1/42 \quad (2,3,7)$.

Pf. Enumerate.

Pf of $84(g-1)$

$\chi(X/G) \leq -1/42$

$(2-2g)/|G| \leq -1/42 \quad \blacksquare$

Pf of $4g+2$

Lemma. $\mathrm{lcm}(p_1, \ldots, p_{n-1}) = |G|$

enumerate.

Thm. Every finite G is a subgp of some $\mathrm{Mod}(S)$.

Pf. Thicken Cayley graph.

Thm. $\mathrm{Mod}(S_g)$ gen. by torsion.

Pf $g \geq 3$. Lantern-nonsep

$T_{d_4} = \prod T_{x_i} T_{d_i}^{-1}$

Ingredient: moving parens.

$T_{x_i} T_{d_i}^{-1} = T_x (f T_x^{-1} f^{-1})$
$= (T_x f T_x^{-1}) f^{-1}$

$\blacksquare$

- Practice asking/answering
- HW 4 Mon.
- OH Tue etc.

10/8/25

Chap 9. Braid gps.

5 defns

1 

2. $\pi_1 \operatorname{Conf}_n \mathbb{C}$ 

3. $\operatorname{Mod}(D_n)$ 

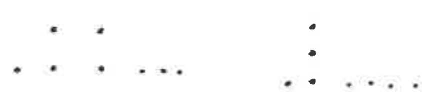
4. $\pi_1 \mathbb{C}^n \setminus \Delta$

5. $A(A_n)$ $\quad \sigma_1$

$1 \leftrightarrow 2 \leftrightarrow 4$ easy

$2 \leftrightarrow 3$
$$\begin{array}{ccc} \operatorname{Homeo}(D^2, P, \partial) & \to & \operatorname{Homeo}(D^2, \partial) \\ & & \downarrow \\ & & C_n(D^2) \end{array}$$

$2 \leftrightarrow 5$ Cell decomp of $\operatorname{Conf}_n \mathbb{C}$
2-cells of dual.



Abelianization

$B_n^{ab} \cong \mathbb{Z}$ Length, #crossings

Torsion free.

Fact. B_n is torsion free.

Pf. finite-dim $K(B_n, 1)$.

Center

$(\sigma_1 \cdots \sigma_{n-1})^n$

Pure braid gp

$1 \to PB_n \to B_n \to \Sigma_n \to 1$

gen by  a_{ij}.

$PB_n \cong PMod(S_{0,n}) \times \mathbb{Z}$

Birman - Hilden thm

$B_{2g+1} \hookrightarrow Mod(S_g^1)$

or $SMod(S_g^1) \cong B_{2g+1}$

$Mod(S_1^1) \cong B_3$

10/13/25

- Practice asking/answering
- OH Tue etc.
- Fri - discussion.

Chap 8. DNB

Thm. For $g \geq 1$.

$\mathrm{Mod}^{\pm}(S_g) \longrightarrow \mathrm{Out}\, \pi_1(S_g)$

is $\cong$.

Injectivity: $K(G,1)$ thy

$\{\text{maps}\}/_\sim \leftrightarrow \{\text{homoms}\}/_\sim$

Surjectivity: main point.

Ingredient #1 Metrics on π_1

QI. The following are q.i.

① Any word metric on π_1
② Any Orbit metric on $\mathbb{H}^2$.

Milnor - Švarc: X proper geod.
G acts pd by isom, cocompact.
Then G f.g. & orbit map is q.i.

Pf. $\lambda = \max d(x_0, s \cdot x_0)$
$r = \inf d(B, gB)$
$K = \max\{\lambda, r\}$, $C = 1$

In partic: automorphisms are qi's of π_1 and $\mathbb{H}^2$.

Ingredient #2. Linking @ ∞

or: homeo @ ∞.

Lemma. $\Phi \in \mathrm{Aut}\, \pi_1(S_g)$ $g \geq 2$.
γ, δ linked $\Leftrightarrow$ $\Phi(\gamma), \Phi(\delta)$ linked.

Pf. Will show not $\Leftrightarrow$ not.
Let K, C qi const. to $\mathbb{H}^2$.
D = diam fund. dom.
$R > 2DK^2 + 2CK$.
$\mathcal{O}_\gamma = \{\gamma^k \cdot x_0 : k \in \mathbb{Z}\}$
- connect dots thru adjacent fund doms, stay in nbd of axis $\gamma \rightsquigarrow \alpha_i$.

Since unlinked, can choose N s.t. each pt of $\mathcal{O}_{\delta^N} \setminus x_0$ has dist $R+D$ from α_i.

Connect by β_i through adj. fund doms. $\boxed{\text{dist} \geq R}$ from α_i.
Length of each edge $\leq 2D$.

If Φ-im's linked
Image edges have length $\leq K(2D) + C$.
$\Rightarrow$ 2 vertices have dist. $\leq K(2D) + C$.

$\rightsquigarrow \alpha, \beta \in \pi_1$

$d(\alpha, \beta) \geq R$

$d(\Phi(\alpha), \Phi(\beta)) \leq K(2D) + C$.

∎

10/15/25

- Practice asking/answering
- Fri/Mon informal, student led

Chap 8. DNB

Thm. For $g \geq 1$

$\mathrm{Mod}^{\pm}(S_g) \xrightarrow{\cong} \mathrm{Out}\, \pi_1(S_g)$

Ingredient #1

$\pi_1(S_g)$ qi $\mathbb{H}^2$.

$\mathrm{Aut}\, \pi_1 \to \mathrm{QI}\, \mathbb{H}^2$

$\to \mathrm{Homeo}^+(\partial \mathbb{H}^2)$

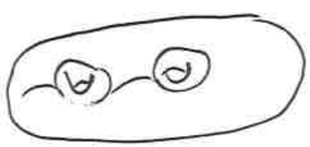
Ingredient #2 Linking @ ∞

Cor. Sides. preserved.

Pf of Thm $\Phi \in \mathrm{Aut}\, \pi_1(S_g)$

Want $f \in \mathrm{Mod}(S_{g,1})$ s.t. $f_* = \Phi$

Choose

c_i 

γ_i 

4 Steps

① $\Phi(c_i)$ simple.

② $\hat{i}(\Phi(c_i), \Phi(c_j)) = 0$ $j > i+1$.

③ $\hat{i} = 1$ $j = i+1$.

④ $\hat{i}(\Phi(c_i), \Phi(c_{i+1}))$ indep of i.

① simple $\Leftrightarrow$ elts of conj class not linked

② $\hat{i} = 0 \Leftrightarrow$ same.

③ $\hat{i} = 1 \Leftrightarrow$ for $\alpha \in a = c_i$, $\beta \in b = c_{i+1}$ linked

the set of conj linked $\beta^k \alpha \beta^{-k}$

④ conjugates lie on same side.

Change of coords

$\rightsquigarrow f$

$\Rightarrow f_*^{-1} \circ \Phi$ fixes the conj. classes c_i. w/orient.

Modify f by conj. to fix γ_i.

γ_1 ✓

$\gamma_2 \mapsto \gamma_1^k \gamma_2 \gamma_1^{-k}$ ✓

$\Rightarrow \gamma_3, \ldots$ already fixed!

Looks like the proof also works with:

10/22/25

- Practice asking/answering
- OH Tue etc.
- HW?

Chap 10. Teich space.

$S = S_{g,n}$

$\mathrm{Teich}(S) = \left\{ \begin{matrix}\text{finite area} \\ \text{hyp metrics}\end{matrix} \right\} / \mathrm{Diff}_0^+(S)$

Two ways

① put a metric on S

② markings

$(S,\varphi_1) \sim (S,\varphi_2)$ if

$$\begin{tikzcd} & S \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] \arrow[d, phantom, "\simeq\ \circlearrowleft"] & \\ X_1 \arrow[rr, "\text{isom.}"'] & & X_2 \end{tikzcd}$$

Length func's

$\ell_X : \mathcal{S} \to \mathbb{R}_+$

$\rightsquigarrow \ell : \mathrm{Teich}(S) \to \mathbb{R}^{\mathcal{S}}$

injective (later!)

Torus

Prop. $\mathrm{Teich}(T^2) \leftrightarrow \mathbb{H}^2$. (unit area Euc.)

Pf. $\mathrm{Teich}(T^2) \to \{\text{marked parallel.}\}$

$\to \mathbb{H}^2$.

Topology

the algebraic topology

$\mathrm{Teich}(S_g) \leftrightarrow DF(\pi_1(S_g), \mathrm{Isom}^+(\mathbb{H}^2)) / \mathrm{Isom}^+(\mathbb{H}^2)$.

$DF \subseteq \mathrm{Isom}^+(\mathbb{H}^2)^{2g}$

Prop. $X \longmapsto \ell_X(c)$ contin.

Dimension counts

① DF

$+6g$ ~~[illegible]~~ $\rho(\gamma_i)$

-3 reln.

-3 conj.

② tiles.

$+8g$ vertices ($4g$-gon)

$-2g$ side lengths

-1 Scale so $\measuredangle = 2\pi$

-3 isometry

-2 pushing base pt.

Pants

Prop. The fn

$\mathrm{Teich}(P) \to \mathbb{R}^3$

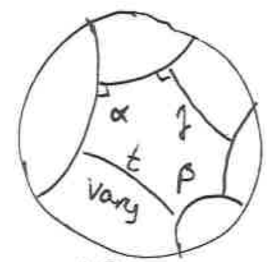
$X \longmapsto \ell(\partial_i)$

is homeo.

Prop. The fn

$\mathrm{Teich}(\mathcal{H}) \to \mathbb{R}^3$

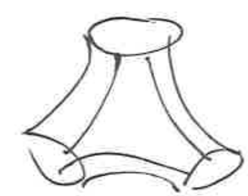
is homeo



$t \to 0 \rightsquigarrow \gamma$ infinite

$t \to \infty \rightsquigarrow \gamma \to 0$.

IVT



- Practice asking/answering
- HW
- OH Tue etc
- FF Seminar.

10/24/25

Chap 10. Teich Space.

Thm. $\mathrm{Teich}(S_{g,n}) \cong \mathbb{R}^{6g-6+2n}$

Dehn-Thurston coords (a warmup)

Fix pants decomp. P, ~~transversely~~

$\delta \leftrightarrow \{(l_i, \theta_i) \in \mathbb{Z}_{\geq 0}^{P} \times \mathbb{Z}^{P} \ \Delta\leq$ on each pants$\}$

Teich (Pants)

Prop The fn

$$\mathrm{Teich}(P) \longrightarrow \mathbb{R}^3_{\geq 0}$$
$$X \longmapsto l_X(\partial_i)$$

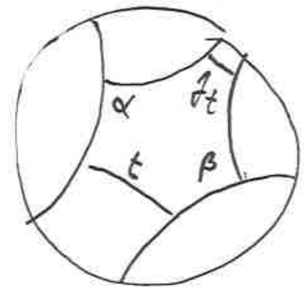
is homeo.

Lemma. The fn

$$\mathrm{Teich}(\mathcal{H}) \to \mathbb{R}^3_+$$

is homeo.

Pf.

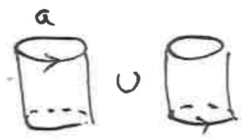


As $t \to \infty$, $\gamma_t \to \infty$.

Pf of Prop. Cut.

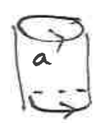
Fenchel-Nielsen coords

Torus Q.



Glue by θ. Are they isometric?

Now



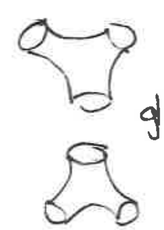
Same Q.

θ small?

once around $\longrightarrow \theta = 2\pi$?

The "twist coord" θ records the shear.

Similar for pairs of pants



glue by $\theta \in \mathbb{R}$

Thm

$$FN: \mathrm{Teich}(S_g) \longrightarrow (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$$
$$X \longrightarrow (l_X(\gamma_i), \theta_X(\gamma_i))$$

is homeo.

Wolpert's formula

$$\omega = \frac{1}{2} \sum \partial l_i \wedge \partial \theta_i$$

indep of choices!

ω = WP form.

$[\omega]$ generates $H^2(MCG)$.

9g-9 thm.

Thm $\exists\ \gamma_1, \dots, \gamma_{9g-9}$

s.t.

$$\mathrm{Teich}(S_g) \to \mathbb{R}_+^{9g-9}$$
$$X \longmapsto l_X(\gamma_i)$$

injective.

10/27/25

- Practice asking/answering
- OH Tue etc.
- Project proposal
- Job talks

Chap 11 Teich geom.

Quasi-conf maps...

$U, V \subseteq \mathbb{C}$ open

$f : U \to V$ smooth off finite set

$\rightsquigarrow df = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}$

$= f_x \, dx + f_y \, dy$

$= f_z \, dz + f_{\bar{z}} \, d\bar{z}$

$f_x = a_x + i b_x$, $f_y = \cdots$

$f_z = \frac{1}{2}(f_x - i f_y)$, $f_{\bar{z}} = \frac{1}{2}(f_x + i f_y)$

$\mu_f = f_{\bar{z}} / f_z$ complex dilat.

f holom $\Leftrightarrow f_{\bar{z}} \equiv 0 \Leftrightarrow \mu_f \equiv 0$.

Also: $|f_z|^2 - |f_{\bar{z}}|^2 = \det$

$\Rightarrow f$ o.pres. $\Leftrightarrow |f_z| > |f_{\bar{z}}| \Leftrightarrow \mu < 1$

$\rightsquigarrow K_f(p) = \dfrac{1 + |\mu_f(p)|}{1 - |\mu_f(p)|}$

$\log K_f(p) / 2 = d_{\mathbb{H}^2}(0, \mu_f(p))$

$K_f(p)$ = ratio of major/minor axes of $df_p(S^1)$.

$K_f = \sup K_f(p)$.

...of Riem. surfs

X, Y Riem. surf's.

$f : X \to Y$ homeo...

f 1-qc $\Leftrightarrow f$ conf.

(removable sing)

Teich problem

Fix $f : X \to Y$ homeo.

Is $\inf_{h \sim f} K_h$ realized? unique?

Yes & yes.

$\rightsquigarrow d_{Teich}(X, Y)$

Measured foliations

Locally 

Prop (Euler-Poincaré)

P_s = # prongs at s.

$2\chi(S) = \Sigma (2 - P_s)$

Measures 

$\rightarrow$ Natural charts

Four constructions

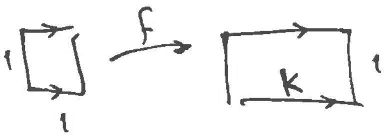
1. Polygon
2. Enlarging a cone.
3. Branched cover
4. Pair of filling (multi-) curves.

10/29/25

- Practice asking/answering
- OH Tue etc.
- Proposals Mon.
- Seminar 4:10.

Chap 10 Teich geom.

Thm (Grötzch)

If f : 

then $K_f \geq K$,

eq. $\Leftrightarrow$ affine.

Pf. $K_f(x,y)$ dilat. $= M/m$

$j_f(x,y)$ jac. $= Mm$

Fact 1. $|f_x(x,y)|^2 \leq K_f(x,y)\, j_f(x,y) = M^2$

Fact 2. $\int_\square |f_x(x,y)|\, dA \geq K$

b/c $\int_0^a |f_x(x,y)|\, dx \geq K$

integr. over y.

Proof of $K_f \geq K$:

$$K^2 \leq \left(\int_\square |f_x(x,y)|\, dA\right)^2 \quad \text{Fact 2}$$

$$\leq \left(\int_\square \sqrt{K_{f(x,y)}}\sqrt{j_f(x,y)}\, dA\right)^2 \quad \text{Fact 1}$$

$$\leq \left(\int_\square j_f(x,y)\, dA\right)\left(\int_\square K_f(x,y)\, dA\right)$$

$$\leq K \cdot K_f.$$

Proof of uniqueness:

Assume $K_f = K = 1$ (WLOG $K=1$)

Want $f = \mathrm{id}$.

All $\leq$ are $=$.

1. f pres. hor, by symm: vertical.
   $\Rightarrow f(x,y) = (u(x), v(y))$
2. x-dir is dir of max stretch a.e.
   Same for y by symm.
   $\Rightarrow |u'(x)| = |v'(y)| \quad \forall (x,y)$
   $\Rightarrow j_f(x,y) = u'(x)^2 = v'(y)^2$
   $\Rightarrow K_f(x,y) = 1$
3. $j_f(x,y)/K_f(x,y) = j_f(x,y)$ const ae.
   $\Rightarrow u'(x)^2 = v'(y)^2$ const. ae.
4. $j_f = 1$ ae. $\Rightarrow f = \mathrm{id}$.

Pf of Teich similar. Need language of...

Measured foliations

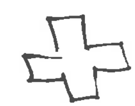
Locally 

measure 

Euler-Poincaré

$$2\chi(S) = \Sigma(2 - P_s)$$

Four constructions.

1. Polygons 
2. Enlarging s.c.c.
3. Branched cover over T^2
4. Pair of filling curves.
5. Quadratic diff.

10/31/25

· Practice asking/answering
· OH Tue etc.
· Proposal Mon.
· FFS

Chap 11 Teich geom.

Last time: meas. fol.

Can have two transv. meas

fol $\rightsquigarrow$ sing. Euc. struc.

$\longleftrightarrow$ holom quad diff

$\rightsquigarrow$ Teich maps 

Lemma. q_Y = hqd on Y cpt.

$h: Y \to Y \quad h \sim id$

$\exists\ M$ s.t. $\forall$ hor α

$$\ell_{q_Y}(h(\alpha)) \geq \ell_{q_Y}(\alpha) - M$$

Pf. · Homotopies move pts bdd amt

· Horizontal $\Rightarrow$ geodesic.

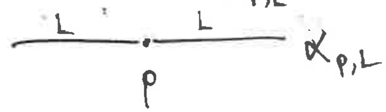
Prop: $h: X \to Y$ Teich map.

w q_X, q_Y, K $f: X \to Y$ $f \sim h$

$$\int_X |f_x|\, dA \geq \sqrt{K}\, \text{Area}\, q_X.$$

Pf. Let $\delta: X \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\delta(p,L) = \int_{\alpha_{p,L}} |f_x|\, dx$$



$$= \ell_{q_Y} f(\alpha_{p,L})$$

Have $\ell_{q_Y}(h(\alpha_{p,L})) = 2L\sqrt{K}$

Lemma $\Rightarrow \ell_{q_Y} f(\alpha_{p,L}) \geq 2L\sqrt{K} - M$

Now:

$$\int_X \delta(p,L)\, dA = \int_X \ell_{q_Y} f(\alpha_{p,L})\, dA$$
$$\geq \int_X (2L\sqrt{K} - M)\, dA$$
$$= (2L\sqrt{K} - M)\, \text{Area}\, q_X$$

Fubini:

$$\int_X \delta(p,L)\, dA = \int_X \left(\int_{\alpha_{p,L}} |f_x|\, dx \right) dA$$
$$= 2L \int_X |f_x|\, dA$$

Combining:

$$2L \int_X |f_x|\, dA \geq (2L\sqrt{K} - M)\, \text{Area}(X)$$

Divide by $2L$ & $L \to \infty$ ∎

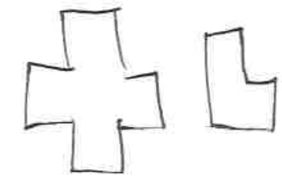
QD's

$$q = \varphi(z)\, dz^2.$$

pos reals $\rightsquigarrow$ hor fol.

neg reals $\rightsquigarrow$ ver fol.

$$\mu(\alpha) = \int_\alpha |\text{Im} \sqrt{\varphi(z)}\, dz|$$



dim QD(X)

$= 3g-3$

$\geq$ Riemann's ineq.

$\leq$ Riemann-Roch.

- Practice asking/answering
- OH Tue etc.
- Projects

11/3/25

Chap 11. Teich geom.

Last time:

pair of transv. mf's

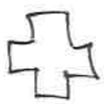
$\leftrightarrow$ sing Euc struc.

$\leftrightarrow$ holom. qd's.

Holom. qd's

In charts: $\varphi(z)\,dz^2$

eg. T^2.



foliations: pos. real

neg. real.

measure: $\int_\alpha |\operatorname{Im}\sqrt{\varphi}|$

$QD(X) \cong \mathbb{C}^{3g-3}$ as v.s.

$\|q\| = \int_X |q|$

dim count

Eul.-Poinc. $4g-4$ simple zeros.

$\rightsquigarrow$ angle 3π

$\rightsquigarrow$ total angle $(4g-4)3\pi$

$(n-2)\pi = (4g-4)3\pi$

$\Rightarrow n = 12g-10$.

$+2(12g-10)$	side vectors
$-\ 12g-10$	match in pairs.
$-\ 2$	last one det.
$-\ 6g-6$	dim of Teich

Teich Existence

$\Omega : QD_1(X) \to \text{Teich}(S_g)$

TET $\Leftrightarrow$ Ω surj.

Prop1 Ω contin.

not obvi!

Prop2. Ω proper.

Inv of domain: inj contin $\mathbb{R}^n \to \mathbb{R}^n$ is open.

$\Rightarrow$ proper inj contin is homeo.

Pf of TET. Ω inj. by TUT

Proper & contin. by Props 1 & 2.

MRM & Continuity of Ω

Factor $QD_1(X) \longrightarrow L^\infty(\mathbb{H}^2) \longrightarrow \text{Teich}$

$\cong$ $\cup$

$T^*\text{Teich} \to T_*\text{Teich}$

Ellipse fields $\leftrightarrow$ functions to $\mathbb{C}$.
in $\mathbb{C}$

In charts: $(-1,1)$ form.

$\mu(z) = \mu(w)\left(\overline{dw/dz}\right)\Big/\left(\frac{dw}{dz}\right)$

$BD(X) = \{\mu : \mu \text{ ess. bdd}\}$

Beltrami eqn: $\mu f_z = f_{\bar z}$

$\mu \in L^\infty(\mathbb{C})$.

Thm $\mu \in L^\infty(\mathbb{C})$ $\|\mu\|_\infty < 1$

$\exists!$ q.c. $f^\mu : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ fixing $0,1,\infty$

with $\mu f^\mu_z = f^\mu_{\bar z}$

f^μ smooth where μ is and varies $\mathbb{C}$-analyt. with μ.

11/5/25

- Practice asking/answering
- OH Tue etc.
- Projects

Chap 12 Moduli space

$\mathcal{M}(S) = \{\text{hyp. str on } S\}/\text{isometry}$
$= \text{Teich}(S)/\text{Mod}(S).$

Stabilizers: Say $f \cdot X = X$

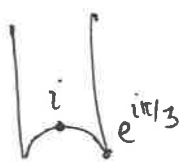
$\Rightarrow f = [\varphi] \quad \varphi \in \text{Isom}(X)$

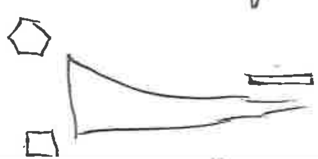
so. $\text{Stab}_{\text{Mod}(S)} X \cong \text{Isom}\, X$

Torus $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \dfrac{az-b}{-cz+d}$

check on $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

Fund dom





~~Propert/disc~~

Length spectrum

$\text{rls}(X) = \{\ell_X(c)\} \subseteq \mathbb{R}_+$

discrete: $\text{rls}(X) \cap [0, L]$ finite.

in fact: $\#\{c : \ell_X(c) \le L\} < \infty$.

Wolpert's lemma

$X_1, X_2 \in \mathcal{M}(S) \quad \chi(S) < 0.$

$\varphi : X_1 \to X_2 \quad K$ q.c.

$\forall c$:

$\frac{1}{K}\ell_{X_1}(c) \le \ell_{X_2} f(c) \le K \ell_{X_1}(c)$

i.e. $d_{\text{Teich}(S)}(X_1, X_2) \le \log K/2$

$\Rightarrow \frac{1}{K}\ell_{X_1}(c) \le \ell_{X_2}(c) \le K\ell_{X_1}(c)$

Pf. Annular cover has modulus

$\pi/\ell_{X_1}(c)$

Grötzch $\Rightarrow$ modulus changes by at most K.

Prop disc

Thm. $\text{Mod}(S_g) \circlearrowleft \text{Teich}(S_g)$ prop disc.

Pf. $B \subseteq \text{Teich}$ cpt

$X \in B$, $D = \text{diam}\, B$

c_1, c_2 fill S

$L = \max \ell_X c_i$

Say $(f \cdot B) \cap B \ne \emptyset$

$\Rightarrow d_{\text{Teich}}(X, f\cdot X) \le 2D$

Wolpert $\Rightarrow \ell_{f(X)} c_i \le \underbrace{K}_{e^{4D}} L$

$\ell_{f^{-1}(X)} c_i \le KL$

$\Rightarrow$ finitely many choices for $f(c_1), f(c_2)$

Alex $\Rightarrow$ finitely many choices for f.

11/7/25

- Practice asking/answering
- OH Tue etc.
- Projects

Chap 12. Moduli space

Mumford's compactness crit

$M_\varepsilon(S) = \ell^{-1}[\varepsilon,\infty)$.

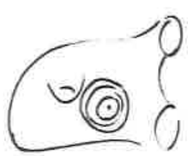
Thm. $M_\varepsilon(S)$ is compact.

Bers constant $\chi(S)<0$. good bound.

$\exists$ $L = L(S)$ s.t. $\forall X \in M(S)$

$\exists$ pants dec with length $\le L$.

Pf.



Pf MCC $g \geq 2$.

$(X_i) \subseteq M(S)$

Want to choose $\tilde{X}_i \in \mathrm{Teich}(S)$ in a FN-box.

Choose $\tilde{X}_i$, subseq s.t. short pants P_i homeomorphic.

New choice s.t. short pants in $\tilde{X}_i$ are all same.

$\rightsquigarrow$ length params bounded.

Now fix twist params.

Isospectral surfs

Thm. $\chi(S)<0$. Fix X

$\#\{Y \in M(S) : rls(Y) = rls(X)\} < \infty$

Pf. Choose $\gamma_1, \dots, \gamma_{9g-9}$

s.t. $\ell : \mathrm{Teich} \hookrightarrow \mathbb{R}^{9g-9}$

For given X & $B_r(X)$

length of γ_i bounded

$\Rightarrow$ finitely many choices.

if $rls(Y) = rls(X)$

$X \in M_\varepsilon$ some ε

$\Rightarrow Y \in M_\varepsilon = \bar{B}$

Thm. $M(S)$ has one end since $C(S)$ conn.

Thm. $M(S)$ simply conn.

Pf. $\mathrm{Mod}(S)$ gen by pt stabs + prop. disc.

11/10/25

- Practice asking/answering
- OH Tue etc.
- Projects.

Chap 12 Moduli space

Thm. $H^*(\mathrm{Mod}(S)) \cong H^*(\mathcal{M}(S))$
over $\mathbb{Q}$.

~~[illegible]~~. Ingr. #1 $H \overset{f.i.}{\trianglelefteq} G \Rightarrow$
same $\mathbb{Q}$ cohom.

Ingr #2. $\exists\ H \trianglelefteq \mathrm{Mod}(S)$
acting freely on Teich

Part III

Chap 13. NTC

Thm. $f \in \mathrm{Mod}(S) \Rightarrow f = [\varphi]$

1. $\varphi^k = \mathrm{id}$ (periodic)
2. $\varphi(M) = M$ M = 1-subman (red)
3. $\varphi(\mathcal{F}_u) = \lambda \mathcal{F}_u \ldots$ pA. $\boxed{\text{mut excl}}$

typical example.

BLM: $\overset{\text{max'l}}{M}$ canonical
$\rightsquigarrow$ decomp, canon. form.

Torus

$SL_2\mathbb{Z} \circlearrowright \mathbb{H}^2$

elliptic $\rightsquigarrow$ periodic (prop disc).
parabolic $\rightsquigarrow$ unique real eigenvec
$\Rightarrow$ one real eigenval $\Rightarrow \lambda = 1$
hyp $\rightsquigarrow$ 2 real eigenvecs
$\rightsquigarrow \lambda, \lambda^{-1} \rightsquigarrow$ Ansov pkg.

or use |trace|: $\{0,1\}$, 2, >2.

3-manifolds

Thm $\chi(S) < 0$

~~[illegible]~~

f per $\Leftrightarrow \widetilde{M}_f = \mathbb{H}^2 \times \mathbb{R}$
f red $\Leftrightarrow M_f$ cont. incomp. torus.
f pA $\Leftrightarrow \widetilde{M}_f = \mathbb{H}^3$

torus case: $\mathbb{E}^3$, Nil, Sol.

Sketch of proof(s).

- Practice asking/answering
- OH Tue etc.
- Projects

Chap 13 NTC

Thm. Every $f \in Mod(S)$ has a rep that is per, red, or pA.

Bers strategy

$Mod(S) \circlearrowleft Teich(S)$

$\tau_f = \inf d(X, f(X))$

$\tau_f = 0$ real $\Rightarrow$ per ✓

τ_f not real $\Rightarrow$ red

$\tau_f > 0$ real $\Rightarrow$ pA

Ingred #1.

Prop 4.1 (BMW) Say τ_f not real. $\tau_f(X_n) \to \tau_f$ then $\pi(X_n)$ leaves every compact set in $M(S)$.

Use: Teich ~~un~~ geod metric sp.
$Mod(S) \circlearrowleft$ prop disc.
f isom.

Pf. Suppose not.

Will find Z s.t. $\tau_f(Z) \le \tau_f$.

$\Rightarrow \pi(X_n) \to \pi(Y)$

$F: M(S) \to \mathbb{R}_{\geq 0}$

$\pi(X) = \min_g \tau_f(g \cdot X)$

F well def:

$\tau_f(gX) = d(gX, fgX)$
$= d(X, g^{-1}fgX)$

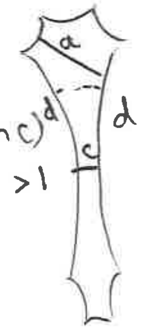
apply proper disc.

F cont.

Ingred #2. Collar lemma.

~~Given X~~ $\exists\ \delta > 0$ s.t.
$\forall S\ \forall X \in Teich(S)$
$\{c \in \mathcal{S} : \ell(c) < \delta\}$ disjoint.

In fact: embedded annulus of large width.



$(\sinh d)(\sinh c) = \cosh(a) > 1$

Pf of reducible case

11/12/25

Choose X_N s.t. $\tau_f(X_n) \lneq \tau_f + 1$

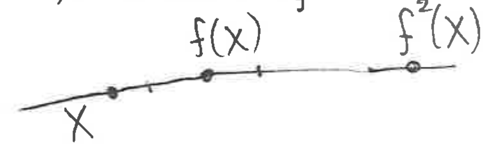
$\ell(X_N) < \left(\frac{1}{K}\right)^{3g-3} \delta. \rightsquigarrow \ell_X(c) \le K \ell_Y(c)$

$\ell_{X_N} c_1$ ~~...~~ $\le \delta$

$\Rightarrow \ell_{X_N} c_1, \dots, c_{3g-3} < \delta$.

Pf of pA case

X realiz. τ_f



$\Delta \le \Rightarrow f^2(X)$ on $\}$

+ uniq geod.

$\rightsquigarrow$ meas fols. pres.

$$\begin{tikzcd}
(f(X), f^*(\mathcal{F}_h, \mathcal{F}_v)) \arrow[r, "h^f"] \arrow[d, "f"] & f^2(X), f^*(\lambda\mathcal{F}_h, \frac{1}{\lambda}\mathcal{F}_v) \arrow[d, "f"] \\
(X, (\mathcal{F}_h, \mathcal{F}_v)) \arrow[r, "h"] & (f(x), (\lambda\mathcal{F}_h, \frac{1}{\lambda}\mathcal{F}_v))
\end{tikzcd}$$

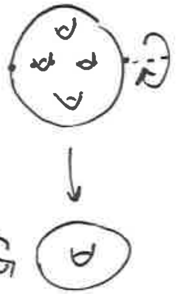
- Practice asking/answering
- OH Tue etc.
- Projects

11/14/25

Chap 14 pA theory

Constructions

① Branched covers



$f \circlearrowright$

f^N has lots of fixed pts

f^{MN} lifts.

② Thurston construction

~~Thm.~~ A, B filling multicurves.

~~Thm.~~ $\rightsquigarrow$ sing Euc struct.

$\rightsquigarrow T_A, T_B$ affine

$\rightsquigarrow \rho : \langle T_A, T_B \rangle \to PSL_2\mathbb{R}$

derivative.

$T_A \mapsto \begin{pmatrix} 1 & -\sqrt{\mu} \\ 0 & 1 \end{pmatrix}$ $T_B \mapsto \begin{pmatrix} 1 & 0 \\ \sqrt{\mu} & 1 \end{pmatrix}$

Thm. A, B filling

$\rho : \langle T_A, T_B \rangle \to PSL_2\mathbb{R}$ deriv.

Then: ① f per, red, pA $\Leftrightarrow$ $\rho(f)$ ellip, par, hyp.

② f red $\Rightarrow$ f conj to T_A^n or T_B^n

③ f pA $\Rightarrow$ $\lambda(f) = \lambda\, \rho(f)$

A, B single curves $\Rightarrow$ $\sqrt{\mu} = i(a,b)$.

In general: PF thm: A primitive.

$\Rightarrow$ A has unique nonneg unit eigenv. which is positive and has λ bigger than all other.

NN^t, N^tN primitive.

Pf. $\rho(f)$ elliptic $\Rightarrow$ isometry.

$\rho(f)$ parab $\Rightarrow$ $\lambda = 1$

take a power so all separ's fixed.

WLOG no accum. pt. $\rightsquigarrow$ closed curves.

cut $\rightsquigarrow$ annuli (no sing).

$\rho(f)$ hyp $\Rightarrow$ pA.

Cor. $\exists$ pA in Torelli.

③ Hom crit.

Thm. $P_f(x)$ sympl. irred over $\mathbb{Z}$

not cyclot.

not poly in x^k

$\Rightarrow$ f pA

④ Kra constr.

γ fill

$\Rightarrow$ Push(γ) pA.

⑤ Braids n prime.

∞ order

cyclic perm

$\Rightarrow$ pA.

11/19/25

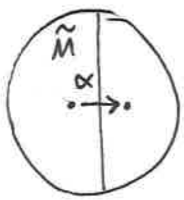
- Practice asking/answering
- OH Tue etc.
- Projects
- Seminar

Chap 14. pA thy.

Thm. α filling $\Rightarrow$ Push(α) pA.

Pf. M = multicurve.

$\rightsquigarrow \tilde{M}$



Thm. $f \in \mathrm{Mod}(S_g)$ pA $\Rightarrow \lambda(f)$ alg int of deg $\leq 6g-6$.

Pf. $k = \#$ sing $\leq 2(2g-2)$

$\Rightarrow \chi(\tilde{S}) \geq 2\chi(S) - k \geq 8-8g$

$\Rightarrow g(\tilde{S}) \leq 4g-3$

$\tilde{\mathcal{F}}_h \in H^1(\tilde{S}) = \mathbb{R}^n$ $\quad n \leq 8g-6$ $\quad$ eigenvector in

$\rightsquigarrow V_+, V_-$ — $\dim \leq 6g-6$.

↳ dim $2g$ $\qquad 2g'-2 = 8g-8$

Fact. If $\deg \lambda(f) > 3g-3$ then even.

Strenner: all other stretch factors arise.

- Explicit construction?
- Most common?
- Torelli etc?

Conj (Fried). λ is a stretch factor $\Leftrightarrow$ λ is bi-Perron.

Thm. $\{\lambda(f) ; f \in \mathrm{Mod}(S_g)\}$ is well order. Pankau, Liechti, Kenyon...

Pf. Eigenval of ~~integer~~ $M \in Sp_{4g-3}(\mathbb{Z})$ + bound.

Homological crit

Thm. $f \in \mathrm{Mod}(S)$

- P_f sympl. irred over $\mathbb{Z}$
- P_f not cyclotomic
- P_f not poly in x^k $\quad k > 1$

$\Rightarrow f$ pA.

$$\mathrm{Sym}(q)(x) = x^{\deg q}\, q\left(x + \frac{1}{x}\right)$$

$\lambda, 1/\lambda$ roots $\Leftrightarrow \lambda + 1/\lambda$ root.

- Symplectic: $A^T J A = J$ or $A^{-1} = -J A^T J$
- Sympl. poly's $\leftrightarrow$ symmetric polys.

11/21/25

- Practice asking/answering
- Projects
- Seminar

Chap 14 pA thy

Lemma. $\mathcal{F}$ pA fol

$\Rightarrow$ no saddle conn, no closed leaves.

Fact. $\mathcal{F}$ pA fol $f \in I$

$\Rightarrow$ $\mathcal{F}$ not or.

Poincaré recurrence

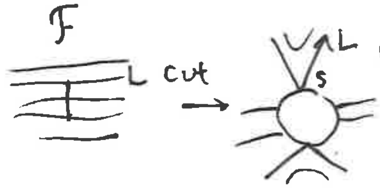
$\mathcal{F}$ = meas fol. any arc int ~~$\mathcal{F}$~~ L int ∞ many times. (L = leaf)

non-example: spiraling, Reeb

Ingred: good atlas

can assume any transv. arc connecting sides has length $\geq \varepsilon_0$

Pf. Suffice to show $\alpha \cap L$ is not a single end pt.



$(\mathcal{F}', \mu')$

assume L does not return to ∂.

Choose good atlas for $\mathcal{F}'$

β = arc of boundary starting at s

$\mu'(\beta) = \varepsilon < \varepsilon_0$

Assume no leaf from β hits a sing. $\rightsquigarrow$ infinite strip injective immersion.

Rectangle decomp

$\mathcal{F}$ = pA fol.

trans arc $\rightarrow$ union of rectangles.

= whole surface, otherwise boundary is a union of closed leaves.

Leaves are dense

τ = arbit. arc.

use rect dec.

You: unique erg.

dense orbits exist

- Practice asking/answering
- OH Tue etc.
- Projects

Chap 14. pA thy.

Thm. f = pA homeo

$\Rightarrow f$ has a dense orbit.

Lemma. $U \neq \emptyset$ open, $f(U) = U$

$\Rightarrow U$ dense

Pf. WLOG f fixes si~~pro~~ngs

Fix s —•— L in F^u

U dense $\Rightarrow L$ contains $x \in U$.

$\rightsquigarrow$ segment $J \subseteq \mathcal{F}^s$, $J \subseteq U$.

Apply $f^k(J)$. $x \to s$

$J \to$ dense leaf

Pf of thm. U_i = countab. basis for S.

$V_i = \bigcup f^n(U_i)$

nonempty, open, f-inv.

Lemma $\Rightarrow V_i$ dense.

Baire cat $\Rightarrow \bigcap V_i$ dense

$\Rightarrow$ nonempty

Any pt in $\bigcap V_i$ has dense orbit. ∎

Thm. f = pA homeo

$\Rightarrow$ per pts dense.

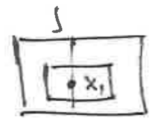
Pf. Choose U 

$V \subset U$ strict.

Poinc. recurr: $\forall N\ \exists\, n > N$

$f^n(V) \cap V \neq \emptyset$.

Let $x_1 \in V$ s.t. $f^n(x_1) \in V$.



WLOG $f^n(J) \subseteq U$.

push onto J

$\rightsquigarrow$ fixed pt x_2.

L = arc of $\mathcal{F}^u$ thru x_2 in U.

WLOG $f^n(L) \supseteq L$.

FT1DD $\Rightarrow f^n(y) = y$.

Ingred #1.

Poinc. Rec. M = finite meas. sp.

$T: M \to M$ meas pres.

$A \subseteq M$ pos. meas. a.e. $x \in A$.

$\exists$ inf seq n_i s.t. $T^{n_i} x \in A$.

Ingred #2. Fund thm of 1D dyn.

$f: [0,1] \to \mathbb{R}$

$\operatorname{im} f \supseteq [0,1]$

$\Rightarrow f$ has fixed pt.

12/1/25

Thm. $f \in \mathrm{Mod}(S)$ pA, $\lambda = \lambda(f)$

a = any hom. cl. scc

ρ = any Riem. metric.

$$\lim_{n \to \infty} \sqrt[n]{\ell_\rho f^n(a)} = \lambda.$$

Pf. Special case: Euc. metric. μ

- upper bound

$$\ell_\mu(f^n(a)) \leq \lambda^n \int_a d\mu^s + \lambda^{-n} \int_a d\mu^u$$

lower: i.e. find a rep.

$$\geq I((\mathcal{F}^s, \mu^s), f^n(a))$$

$$= \lambda^n I((\mathcal{F}^s, \mu^s), a)$$

(one direc.)

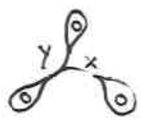
Same for: word length

int. nums.

12/3/25

- Practice asking/answering
- Projects

Chap 15. Thurston's proof.

$\rightsquigarrow$ 

$\longrightarrow \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

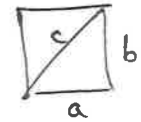
measured train tracks $\longleftrightarrow$ measured foliations.

$$\mathcal{T}(S) \hookrightarrow P\mathbb{R}^{\mathcal{S}}$$
$$P\mathcal{MF} \nearrow$$

Thurston: $\mathcal{T}(S) \cup P\mathcal{MF}(S) \cong B^{6g-6}$

and $Mod(S)$ acts by PL-homeo on S^{6g-7}

Brouwer $\Rightarrow$ fixed pt $\rightsquigarrow$ cases.

Torus case 

$$\mathcal{T}(S) \hookrightarrow P\mathbb{R}^{\{a,b,c\}}$$
$$P\mathcal{MF} \nearrow$$

Can do computations!

3 train tracks