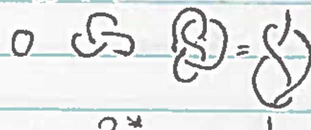
Math 150
Knot Theory

Fall 2009
Tufts U

R. Thomson (Lord Kelvin): atoms $\longleftrightarrow$ knots in aether

Tait: "periodic table" 

Q. What are all knots?*

(take intuitive def of knot for now: string...)

e.g. Fig 1.4 in Adams. Which knot is this?

better: 

Q. How do we know when two knots are same?

Or: when is a knot really knotted?

Ex. 1.3 in Adams.

*Try to list all knots
0 crossings (1 diagram)
1 crossing (4 diagrams)
2 crossings (HW).

Fig 1.29 in Adams: even if I tell you it is unknotted, it might be hard to do!

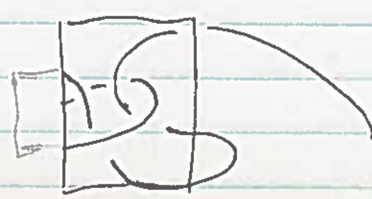
How (not) to define a knot

$f: S^1 \to \mathbb{R}^3$ (image of), continuous, injective.

Not enough! Wild Knots:

do first $\to$ 

(Antoine's Necklace)

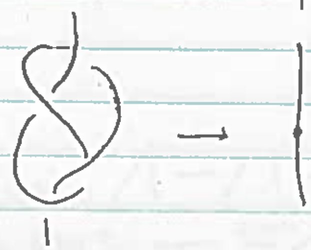


unknotted at each finite stage!

Fig 1.15 in Cromwell

Solution: PL knots. set of pts $\rightsquigarrow$ closed polygonal curve
simple
knot: simple closed polygonal curve in $\mathbb{R}^3$
unknot: 3 noncolinear pts.

Equivalence of knots
"Continuous" again bad



"bachelor's unknotting"

Elementary deformation

only intersection is $[p_1 p_n]$

Equivalence (check this is an equivalence relation)

Convex knot in plane $\Rightarrow$ unknot
All unknots are equivalent.

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Last time: Cool - knots are fun! 
Eww - knots are icky! 
Blech - knots are clunky 

For HW: Introduce pretzel knots, Whitehead link.

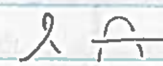
Def of knot, equivalence
"knot" usually means equiv. class

Clunky: moving a vertex takes 2 moves 
deleting an unnecessary vertex takes 3: 
translating a knot even worse!

We want to go back to drawing nice pictures, but we need to prove a few facts
1. All knots have diagrams
2. Two knots with same diagram are equivalent
3. Any two diagrams for same knot differ by a sequence of Reid. moves.

Q What basic moves are needed?

example: 



anything else?

Diagrams

Projection $\mathbb{R}^3 \to$ plane (nearest point)

A projection of a knot 

Diagram: A picture of a knot obtained from a projection, where parts of the projection are removed, to indicate undercrossings.

Def: A vertex is a defining pt that is not colinear with its two neighbors

equivalent: {vertices of a knot} is a defining set with property that no subset defines same knot

Def: A knot projection is regular if no 3 pts project to same point & vertices do not project to same point as any other pt.

<u>Fact 1</u>: K knot determined by $(p_1, \dots, p_n)$

$\exists\ \delta$ s.t. if $d(p_i, p_i') < \delta\ \forall\ i$

then the knot determined by (p_i) is equiv to K.

$\boxed{\text{Equiv class of } K \text{ open in } \mathbb{R}^{3n}}$

Cor: Translate, Rotate knots.

<u>Pf</u>: A metric neighborhood of a line segment is a "capsule"

Choose a metric nbd of K so that only adjacent capsules intersect.

+ each ball in exactly two capsules

For each edge, look a distance to set of nonadjacent edges

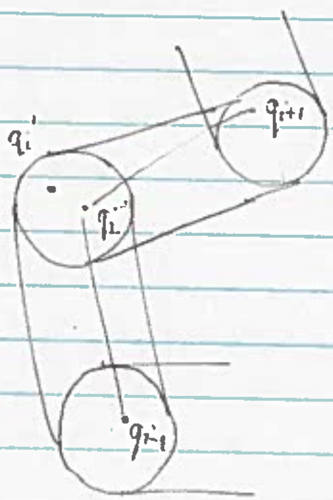
compactness, positivity $\Rightarrow$ min $= \delta$.

+ for each vertex, look at distance to nonadjacent edges.

Check that δ works.

Let q_i be any where in δ ball about p_i.

To show: can move q_i to q_i' in δ-ball about p_i



Draw triangle q_i, q_i', q_{i+1}

or q_i, q_i', q_{i-1}

At most one is bad:

in sense that $q_i q_{i+1}$ or $q_{i-1} q_i$ runs through it.

No other edge can run through by way we chose capsules.

Say q_i, q_i', q_{i+1} works

Now draw triangle q_i, q_i', q_{i-1} to get rid of q_i

Lies entirely in capsule, by convexity

q_{i+1} outside capsule

$\Rightarrow$ q_{i+1} not in triangle

$\Rightarrow$ $q_{i+1} q_i$ not in triangle

$\rightarrow$ can do final move.

<u>Cor</u>: Translate, rotate, scale, etc.

Fact 2 K determined by $(p_1, \dots, p_n)$

$\forall \delta \ \exists K'$ determined by (q_i)

so $d(p_i, q_i) < \delta$, $K' \sim K$,

projection of K regular

Fact 1 + Fact 2

$\Rightarrow$ regular projections

Pf: A projection is regular if

① No segment is vertical

② No vertices span a plane containing vertical line

③ No triple points

Rotate to get rid of 1, 2, then deal with 3.

Fact: Knots w reg proj, equiv to K open in $\mathbb{R}^{3n}$

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To turn knot theory into a "game", need 3 facts:

1. Every knot has a diagram
2. Equal (equivalent) diagrams correspond to equivalent knots
3. Any two diagrams for same knot differ by certain "moves"

Fact: Every knot has a diagram

Pf: Think of sphere is a "track ball". Show "bad set" is a finite union of points and great circles.

Bad set:
1. Vertical segment
2. 3 vertices span a vertical plane.
3. 3 points of different segments lie on a vertical line.

1 is finitely many points

2 is finitely many great circles

3 is finitely many points (lin alg exercise)

Thus, can rotate any knot by arbitrarily small amount to get a ~~regula~~ diagram. Use "Wiggling Knots"

Equivalence of diagrams

Elementary moves on diagrams

Do a move like before, but inside triangle contains no vertices or crossings.

Wiggling diagrams

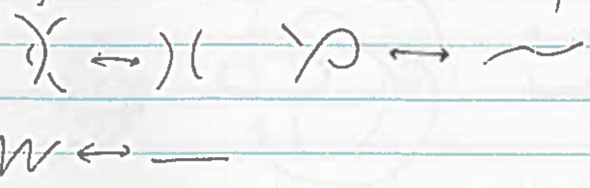
Fact: Given a diagram D of a knot

$\exists$ $\delta > 0$ s.t. if you move vertices a distance at most δ, diagram is equivalent

Fact: Given a knot and a diagram, $\exists\, \delta$ s.t. can rotate the knot by δ so diagram does not change.

Moves on Diagrams.

In unknotting a standard unknot, we already saw the moves



Also need: 

Will prove what all moves are. Idea: At most one crossing/vertex "in" each move

See we need 

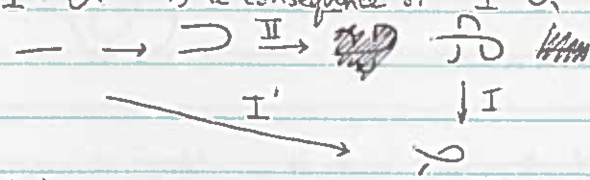
Only one version of each move except "fish move" & "hazardous waste move"

Two versions of former 

Four of latter 

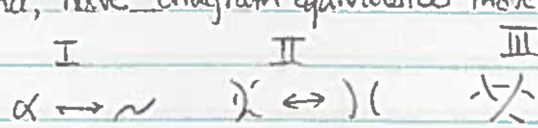
note: left two are same & right two are same, so only two.

Surprise: I':  is a consequence of I and II



Also: III'  is a consequence of II and III  (exercise)

So, in end, have diagram equivalence moves plus



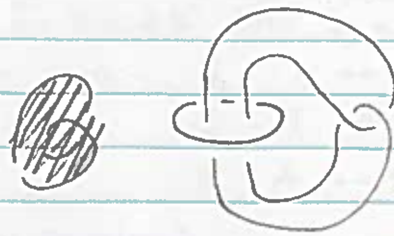
"Reidemeister Moves"

Q. Are all 3 needed?

Reidemeister Theorem: Any two diagrams of equivalent knots differ by a finite sequence of Reidemeister moves.

From HW: Symmetric pics of Whitehead Link:

Morgan



twist inner loop, then twist outer

Jon
Sarah
Eric

Others:

Ryan
Matt

Mike

not symmetric

Key observation:

unknotted.

For pretzel knot symmetry problem, think of building the knot using triangular prism template.

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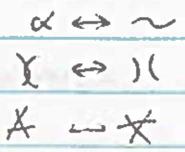
Reidemeister's Theorem If K and K' are equivalent knots, then any two diagrams for K and K' differ by a finite sequence of Reidemeister moves: I, II, III



Instead of "differ", say "can be made equivalent"

(plus diagram equivalence).

(Other direction for HW).

Lemmas Fact 0: Can rotate knots.

Fact 1: Given any knot, and any $\epsilon > 0$, there is a rotation of the knot, with angle of rotation $< \epsilon$ so that the projection is regular.

Fact 2: Given a knot, and any projection, there is a $\delta > 0$ s.t. if the vertices of the knot are all moved a distance $< \delta$, then the resulting diagram is equivalent.

Proof of Thm Assume diagrams are both obtained via same (vertical) projection.

K equiv. to K' means there is a sequence

$$K = K_1, \dots, K_n = K'$$

where K_i differs from K_{i+1} by an elementary deformation.

Rotate projection direction slightly st. each K_i has regular projection & diagrams for K, K' don't change (Facts 1 & 2).

We now look at a single elementary deformation and show it is a composition of R moves.

Idea: break this one move into a sequence of small moves:

subdivide

First order of business: get rid of

Next: Subdivide so in each triangle, only see:

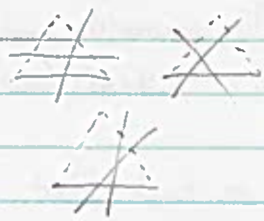
(1) one crossing
or (2) one vertex
or (3) one edge, no vertex
or (4) nothing.

Also: Observe any piece of the knot whose projection lies in the triangle either lies totally above or totally below

Type (1)

Type (2)

all variations are type III



Type (3)

Type (4)

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Theorem: There exists a nontrivial knot

Idea: coloring

A coloring of a ~~knot~~ diagram is an assignment of red, green, or blue to each arc of a diagram, so each crossing has one color or three colors.

A ~~knot~~ diagram is colorable if it has a ~~diagram~~ nontrivial coloring.

 colorable  not

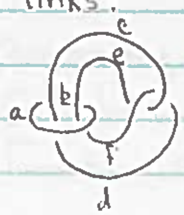
So trefoil is different from unknot. Wait a minute... need to show that colorability is a property of the knot, not the diagram.

Prop: If one diagram of a knot is colorable, then all of them are

Pf: Any two diagrams differ by R moves, so suffices to do a local check.

Thus the trefoil knot is knotted!

What about links? Is, say, the Whitehead link colorable?



a = red
b = green
⇒ c = blue
⇒ d = green
⇒ e = blue
⇒ f = blue

a = red
b = red
⇒ everything red.

But wait! Unlink on two components is colorable so Whitehead is really linked. (Note: Prop works for links)

Prop: Any two diagrams of the same knot have the same number of colorings

Pf: Let D, D' be two diagrams for K

Choose a seq of R moves $D \to D'$

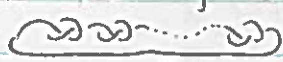
$\rightsquigarrow$ function: (colorings of D) $\longrightarrow$ (colorings of D')

Inverse sequence of R moves gives inverse.

An invertible function ~~from~~ between finite sets is a bijection. ☒

Theorem: There are infinitely many distinct knots.

Pf: We will show by induction that

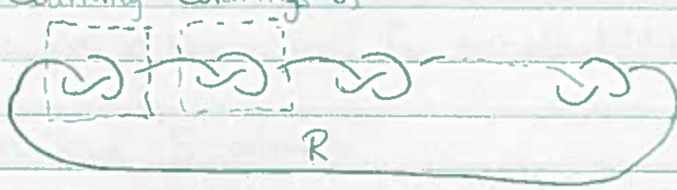


has $\boxed{3 \times 2^k}$ colorings.

Base case is trefoil.

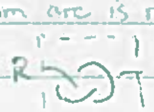
How to find all colorings? Generalize?

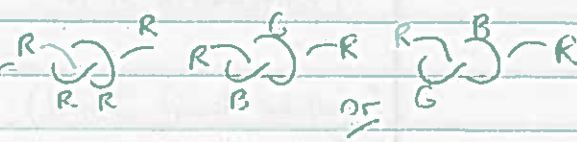
Correction: Counting colorings of



^ ~~connecting arcs~~

Say bottom arc is red.

First box: 

either 

Thus: • "connecting arcs" all red

• 3 ways to color each box

$\Rightarrow 3^k$ colorings

$- 1$ trivial coloring

Multiply by 3 since bottom arc could be blue or green

$3(3^k - 1)$

Coloring with more colors 9/24.

$\mathbb{Z}/p\mathbb{Z}$ is the set $0, \dots, p-1$ with (mod p) arithmetic.

Take p prime $\Rightarrow$ multiplicative inverses (check for $p=5$)

A p-coloring of a knot diagram is an assignment of an element of $\mathbb{Z}/p\mathbb{Z}$ to each arc so the following holds:

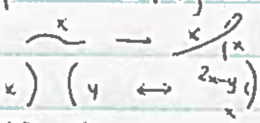
$x \equiv y + z \pmod{p}$.

+ At least two colors

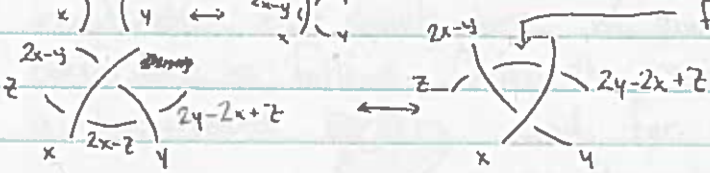
Check: 3-coloring in this sense is same as before.

Prop: p-colorability is a knot invariant

Pf:



Note:  $\Rightarrow x \equiv y$



forced to be: $4x-2y-z$

$2x-(2y-2x+z)$

same!

note: other crossings unaffected

if started w/ at least 2 colors, still have at least 2. ∎

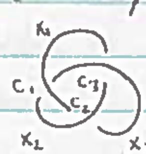
This gives a cleaner proof for tri-colorability (no cases).

Is this knot 5-colorable?

(not best example, since it is alternating).

(John C found solution.)

Hard to guess. More systematically, write a system of linear equations mod p:



$$\begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ \hline c_1 & & & \\ c_2 & & & \\ c_3 & & & \end{array}\quad \begin{array}{c} \\ c_1 \\ c_2 \\ c_3 \end{array}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

$\exists$ p-coloring $\Leftrightarrow$ (mod p) sol'n to $Mx = 0$

$\Leftrightarrow \det M \equiv 0 \pmod{p}$

Proof of $\exists$ sol'n $Mx \equiv 0 \pmod{p} \Leftrightarrow \det M \equiv 0$ is Gauss-Jordan elimination, which still works since $\mathbb{Z}/p\mathbb{Z}$ has multiplicative inverses.

Problem: $(1,1,\dots,1)$ is nontrivial in terms of linear algebra, but trivial for us.

To address this problem, we use the fact that we can make the first arc any ~~c~~ color we want.

In linear algebra terms: Note $(1,\dots,1)$ is always a solution
(sum of columns is zero)

Sum of any two solutions is another sol'n
(linear subspace).

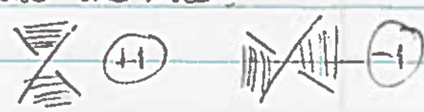
$\Rightarrow$ take $x_n = 0$, or delete last column

Now any nontrivial lin. alg. solution gives a nontrivial coloring.

But we lost the squareness.

Fix: show we can delete any one row, by finding a linear combination that equals zero. We give rule for coefficients as follows. Color the knot diagram as a checkerboard

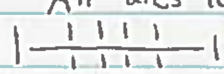
and, for each crossing, use the rule:



Claim: When we add rows using these coefficients, we get 0.

Pf of claim: Suffices to check one column (= arc) at a time.

All arcs look like



or

overcrossings: ± 2 unders: ± 1

color & check.

So we get an $(n-1)\times(n-1)$ matrix. The absolute value of this matrix is called the determinant of the knot (or link). ~~The knot~~

For HW: mod p nullity and determinant are knot invariants.

Thm: Whitehead Link is not equivalent to Mike's Whitehead Link

aka Solomon's Seal

Pf: Det = 8 Det = 4

9/29.

EVOLUTION OF AN INVARIANT

- Tri-colorability (yes/no)
- Coloring numbers
- Determinant
- Alexander polynomial — distinguishes all knots with ≤ 8 crossings!
- more...

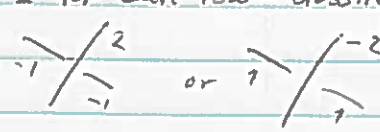
example: 4_1 & 5_1 (fig 8 knot & (5,1)-torus knot)
both have det = 5   det forgets too much info
$4_1 \rightsquigarrow t^2-3t+1$
$5_1 \rightsquigarrow t^4-t^3+t^2-t+1$

First: new proof that rows of coloring matrix add to 0, with correct sign choices.

Step 1 – Fix a checkerboard coloring of knot projection.
Proof that you can do this: draw a vertical line to left of knot. Obviously, can color everything to the left of the line. Move line to right. Critical pts look like

(Can assume general pos).

Step 2 – Choose ± for each row = crossing. That is, must choose



at each crossing.

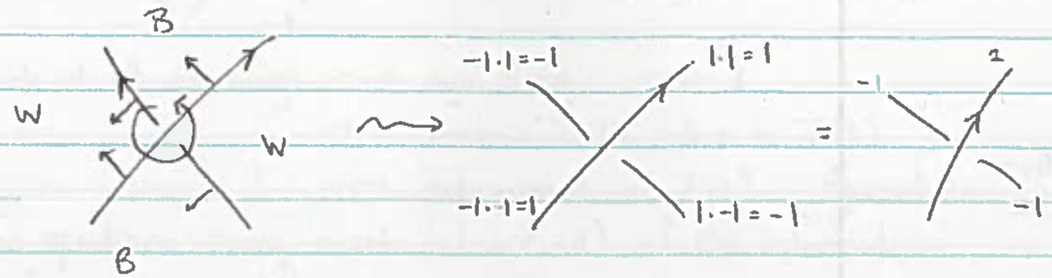
Step 3 – Orient the knot diagram. At each of 4 (half-)arcs at a crossing, draw a little arrow using RH rule.

Assign: ± 1 depending on whether little arrow points to black or white region

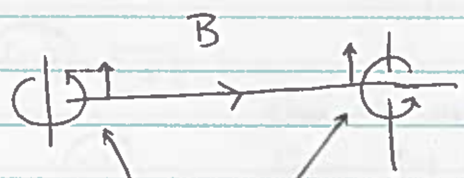
Say: black $= +1$ white $= -1$

± 1 depending on whether little arrow agrees with counterclockwise curl at the crossing

Multiply these two that gives the labelling



Step 4: Check that sum of rows = crossings is 0.
Even better: adjacent half-arcs cancel.



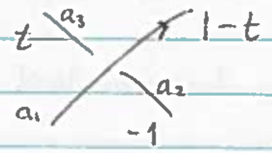
look here and here.
Colors agree (both point to black) but curls disagree. ▨

Alexander Polynomial

{knots} $\longrightarrow \mathbb{Z}[t,t^{-1}]$ = "Laurent polynomials in t"

Step 1: Orient the knot

Step 2: At each crossing, assign labels to arcs



$$(1-t)a_1 = a_2 - ta_3$$

Look at all crossings $\rightsquigarrow$ sys of eq's in a_i, t

$\rightsquigarrow$ matrix with entries in $\mathbb{Z}[t]$

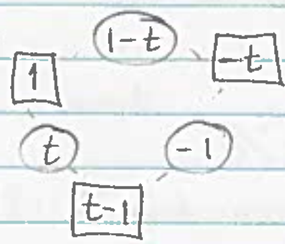
$\rightsquigarrow$ determinant in $\mathbb{Z}[t]$ Equivalence: mult. by $\pm t^k$

The equations carry much (almost all) of the information of the original diagram. Example: recover the diagram giving the Alex matrix:

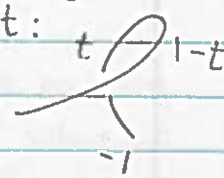
$$\begin{bmatrix} 1 & -1 & 0 \\ 1-t & -1 & t \\ -t & 0 & t \end{bmatrix}$$

(Note: A=B ≈ A=Ɛ for example) what else?

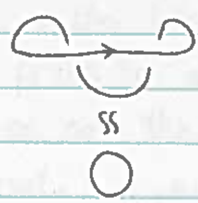
Notice:



that is an entry of 1 comes from $1-t$ & t:



Answer:



det: $-t \equiv 1$

det = 1

Notation

$\Delta_K(t)$

Check Alex. poly for trefoil: $t^2 - t + 1$

Note: Best rep. for $\Delta(t)$ is the unique one with a constant term.

Fact: $|\Delta_K(-1)| = \det(K)$ (can check locally)

Fact: $\Delta_K(t) \equiv \Delta_{\bar{K}}(t^{-1}) \equiv \Delta_{K^{-1}}(t^{-1})$

Fact: $\Delta_K(t) \equiv \Delta_K(t^{-1}) \Rightarrow \Delta_K(t)$ palindromic.

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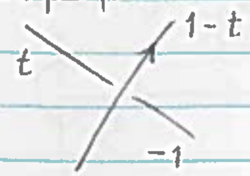
Recall def of Alex. poly. :

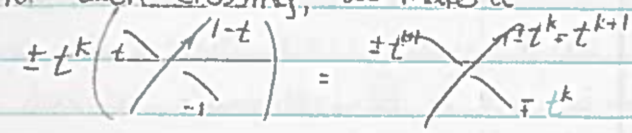


$\Delta_K(t)$

<u>Thm</u> $\Delta_K(t)$ is a knot invariant.

<u>Lemma</u>: Let $\{c_i\}$ = rows of Alex matrix = crossings.
$\exists\ j_i, k_i$ s.t. $\sum (-1)^{j_i} t^{k_i} c_i = 0$.

<u>Pf</u>: In other words, for each crossing, we make a choice of



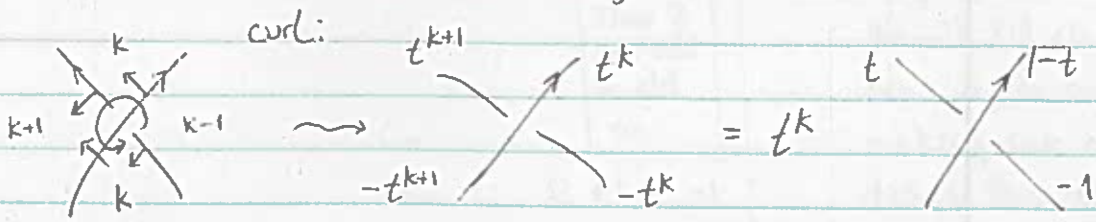
so that sum is zero.

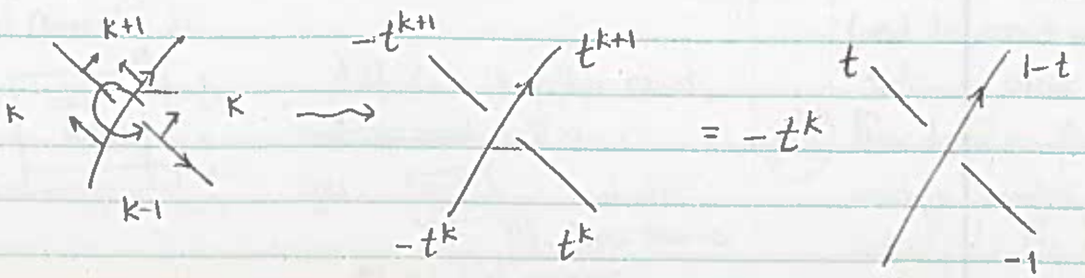
Step 1: Label each region of diagram by an integer using the rules ① Outside is 0
② k+1 ↑ k

Note: this is the winding number

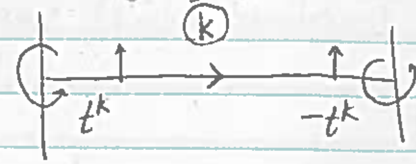
example:

Step 2: At each crossing draw 4 arrows according to RH rule. Label arcs by $\pm t^k$ where k is the label of the region that the arrow points to and the sign is given by whether or not the arrow agrees with counterclockwise curl:





Step 3: Check everything cancels



labels agree
curls disagree ▨

Linear Algebra Fact: If rows & cols of an $n \times n$ matrix all add to zero, then ~~any~~ all $(n-1)\times(n-1)$ minors have same determinant.

Pf: Start by taking out last row, last col. (old minor)

We'll show you can swap last col for next-to-last col. (new minor).

Here's how: Starting with $(a_1 \cdots a_n)$

Replace a_{n-1} with $a_1 + \dots + a_{n-1} = a_n$

This does not change the det of the new minor,

but now the new minor is same as old minor ▨

Lemma + Fact $\Rightarrow$ all $(n-1)\times(n-1)$ minors of Alex. matrix have same det.

Proof of Theorem

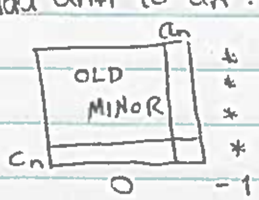
Need to show invariance under Reidemeister moves.

New matrix:

$$\begin{array}{c|cccc} & a_1 & \cdots & a_n & a_{n+1} \\ \hline c_1 & & & \multicolumn{2}{c}{\text{These 2}} \\ \vdots & & & \multicolumn{2}{c}{\text{cols add}} \\ & & & \multicolumn{2}{c}{\text{to old}} \\ c_n & & & \multicolumn{2}{c}{a_n} \\ c_{n+1} & 0 & \cdots\ 0 & +1 & -1 \end{array}$$

Any minor will do.
We will kill a_n & c_n.
Now we do row/col ops, making sure not to disturb this determinant.
(need to avoid adding a_n/c_n to other rows/cols).

Add a_{n+1} to a_n.



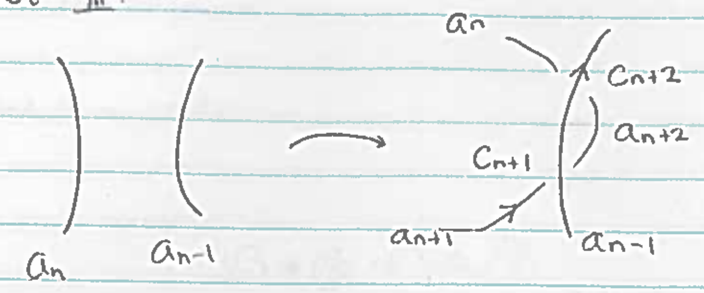
Add c_{n+1} to other rows, ~~end up with~~ kill a_n, c_n.

get  which has same det as old minor.

NB After doing row/col ops, matrix might not satisfy LinAlg fact

Other moves are similar, and are detailed on p. 64-66 of Reidemeister (English translation). We'll show II but not III.

II



New matrix:

$$\begin{array}{c|ccccc}
 & a_1 \cdots & a_{n-1} & a_n & a_{n+1} & a_{n+2} \\ \hline
c_1 & \multicolumn{2}{c}{\text{Same as}} & \multicolumn{2}{c}{\text{Sum to}} & 0 \\
\vdots & \multicolumn{2}{c}{\text{old}} & \multicolumn{2}{c}{\text{old}} & \vdots \\
c_n & \multicolumn{2}{c}{\text{matrix}} & \multicolumn{2}{c}{a_n} & 0 \\
c_{n+1} & & 1-t & & t & -1 \\
c_{n+2} & & 1-t & t & & -1
\end{array}$$

Subtract c_{n+2} from c_{n+1}
Add a_{n+2} to other cols
Add c_{n+1} to other rows

Get

$$\begin{pmatrix} \boxed{\text{old matrix}} & & \\ & \boxed{t} & \\ & & \boxed{1} \end{pmatrix}$$

III exercise.

Example: (2,n) torus knots are distinct. Minor looks like.

$$\begin{pmatrix}
1-t & -1 & 0 & \cdots & & 0 \\
t & 1-t & -1 & 0 & \cdots & 0 \\
0 & t & 1-t & -1 & 0 \cdots & 0 \\
 & & \vdots & & & \\
0 & \cdots & & 0 & t & 1-t
\end{pmatrix}$$

Highest term: t^{n-1}
Lowest term: 1

10/6

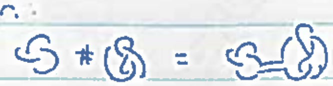
Gems so far:

1. Reidemeister's Thm
2. Alexander Polynomial

Next

3. Prime decomposition

Connect sum operation:



(we've seen this before: $\#_n$ 

Note O acts as identity element, associativity holds, inverses?

Convince yourself # is well-defined

mirror?

A knot K is prime if whenever $K = K_1 \# K_2$, at least one K_i is trivial.

Theorem: Any knot can be written uniquely as a connect sum of primes.

Theorem: A nontrivial knot has no inverse under connect sum.

Proofs will use theory of surfaces.

Surfaces

A surface is something that locally looks like $\mathbb{R}^2$ (or upper half-plane)

Examples: sphere, disk, $\mathbb{R}^2$

Nonexamples: , Knots

More examples: Torus 

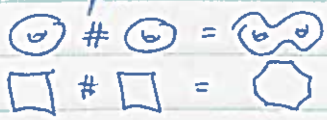
Möbius strip 

Klein Bottle

Real proj. plane

Orientable vs. Nonorientable

Connect sum operation:



Define combinatorial surface in terms of triangles.

Classification of Surfaces
(Following Conway)

Six Lemmas

① Connect sum with S^2 does nothing

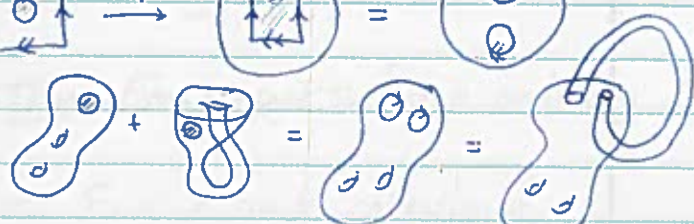
$$S \# S^2 \approx S$$

② Connect sum with $T^2 \leftrightarrow$ adding a "handle"

③ Connect sum with KB $\leftrightarrow$ adding a "cross handle"

first understand $KB - D^2$:

now:



④ If S is nonorientable, then

$$S \# T^2 = S \# KB$$

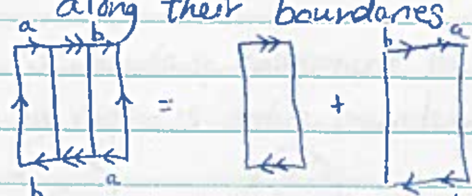
Can't tell difference

between  & 

⑤ $\mathbb{RP}^2 \# \mathbb{RP}^2 \approx KB$

Step 1: $\mathbb{RP}^2$ − disk $\approx$ Möbius strip

Step 2: KB = two Möbius strips glued along their boundaries



⑥ Connect sum with $\mathbb{RP}^2 \leftrightarrow$ adding a "crosscap"

Theorem: Every connected compact surface is equivalent to one of the following:

S^2

$T^2 \# \cdots \# T^2$

$\mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$

or one of these minus a finite number of open disks.

Equivalent formulation: Call a surface <u>ordinary</u> if it is obtained from a collection of spheres by adding handles, adding crosshandles, adding crosscaps, and removing disks.

<u>Thm</u>: Every compact surface is ordinary.

<u>Proof</u>: By assumption, S is a finite collection of triangles, glued together. Start with a pile of triangles, and do gluings one by one.
First triangle is ordinary. Now check that each successive gluing results in an ordinary surface.

<u>Case 1</u> Glue one entire boundary component of S to one other entire boundary component

handle   cross handle   gluing two pieces

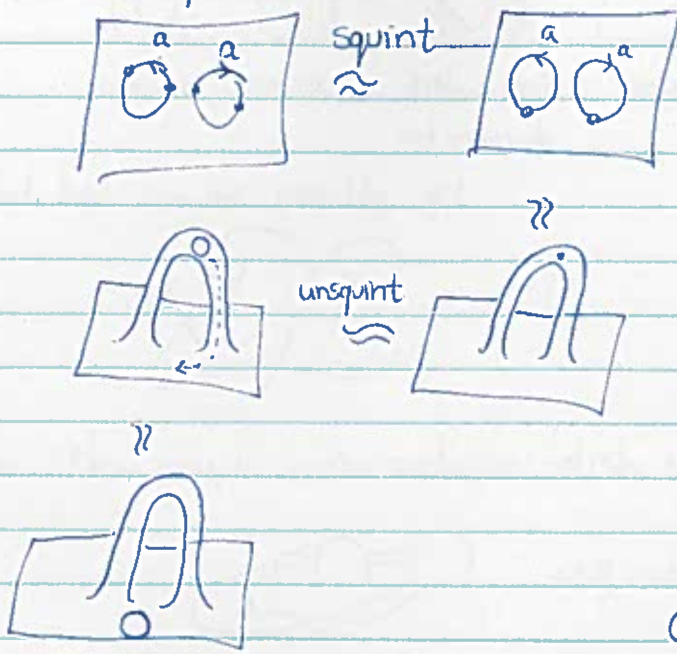
<u>Case 2</u> Glue a boundary component to itself (gluing region is entire boundary component)

cap   crosscap

Case 3 Glue part of one boundary component to all or part of another, or same, boundary component.

Idea: Reduce to Cases 1 & 2 by "squinting", or stretching the boundary components so it looks like gluing region is entire boundary component.

One example:



QED.

Why are we studying surfaces again? 10/15

Knots – in particular prime decomposition

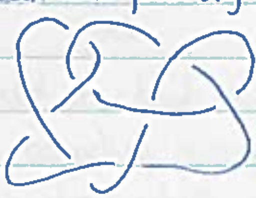
Theorem (Seifert) Every knot is the boundary of a surface embedded in $\mathbb{R}^3$. What is more, we can choose the surface to be orientable.

Def: An orientable surface is in the theorem is called a Seifert surface.

examples:

unknot — unlink — trefoil, but not orientable — unknot

But how can we possibly get:



???

Classification of Surface says the Seifert surface is one of:

Idea. New way to make surfaces: disks + bands

still unknot...

Which surfaces are these?

pair of pants — one ∂ comp. can find curves that intersect once... — same as previous

(still unknot)

Proof of Theorem

Step 1: Orient the knot projection

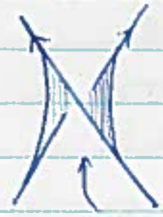
Step 2: Surger 

Result is an oriented 1-manifold, hence it is a collection of disjoint oriented circles.

Step 3: Lift circles so innermore ones are higher

Step 4: Fill in circles with disks

Step 5: Add one band for each crossing.



band.

Now we need to check the resulting surface is orientable (easy to see boundary is original knot).

Color top of disk white and bottom black if boundary is clockwise, top black bottom white counterclockwise.

Color bands by extension. Check consistency.

Two cases:  

∎

Can do with above example, or trefoil

Again, we know this surface is one of 

How do we know which one?

Euler Characteristic

Take a surface ~~with~~ and cut it into polygons

$V = \#$ vertices $\quad E = \#$ edges $\quad F = \#$ faces

The Euler characteristic is $\chi = V - E + F$.

We will see: Thm: χ is an invariant of the surface

example: S^2

	V	E	F
tetrahedron			
cube			
octahedron			
icosahedron			
dodecahedron			

note duality

$$\boxed{V - E + F = 2}$$

$\chi(T^2) = 0 \qquad 1-2+1$

$\chi(S - \text{disk}) = \chi(S) - 1$

$\chi(S_1 \# S_2) = \chi(S_1) \# \chi(S_2) - 2$

$\Rightarrow \chi(T^2 \# T^2) = -2$

$\chi(S_{g,b}) = 2 - 2g - b$

$\chi(\text{Disks} + \text{Bands}) = D - B$

So trefoil surface above has $\chi = 2 - 3 = -1$

10/20

Goal: Factor knots

Recall defn of Seifert surface

Def: The genus of a knot is the minimal genus of a Seifert surface (invariant by defn)

Fact: $\text{genus}(K) = 0 \Leftrightarrow K = 0$.

Thm: $\text{genus}(K \# K') = \text{genus}(K) + \text{genus}(K')$

Lemmas: Let $S = S_{g,b}$ $S_c = S$ cut along c.
- c nonsep scc $\Rightarrow \text{genus}(S_c) = g-1$
- c sep scc $\Rightarrow \text{genus}(S_c) = g$
- c sep arc $\Rightarrow \text{genus}(S_c) = g$

(genus of disconn. surface is sum of genera).

Pf: Use χ.

Proof of Thm (≤) Glue (minimal) surfaces for K, K' to get some, not necessarily minimal, surface for $K \# K'$

(≥) Start with {minimal} surface for $K \# K'$, want to show you can decompose it into two not necessarily minimal surfaces for K, K'.

To this end: Let Σ be a 2-sphere decomposing $K \# K'$ into K & K', and let S be a Seifert surface for $K \# K'$. (assume minimal)
$S \cap \Sigma$ is a collection of disjoint circles and one arc (after moving into general position). Use that $K \# K' \cap \Sigma = 2$ points.
Use lemma to "surger away" circles of intersection, starting innermost first. For instance, if some circle is nonseparating in S, then surgery  yields a smaller genus Seifert surface for $K \# K'$.
At end, have one arc of intersection, so surface for $K \# K'$ splits up as desired. Sum of genera of pieces is genus of big surface, by lemma. ▨

Cor: Knots do not have inverses

Another proof (?): Say $K \# K' = O$

$$O \# O \# \cdots = O$$
$$(K\#K')\#(K\#K')\# \cdots = O$$
$$K\#(K'\#K)\# \cdots = O$$
$$K \# O = O$$
$$K = O.$$

Can we make this work?
Yes. See Thm 1.5 in Prasolov - Sossinsky

Cor: Genus 1 knots are prime.

Cor: Knots have prime decompositions.

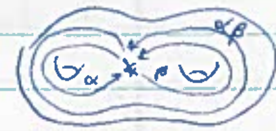
Uniqueness is a little harder. See p. 19 of Lickorish.

10/27

So far: Knots $\to$ determinants
$\to$ polynomials

Next: Knots $\to$ groups

Fundamental group of a topological space:
elements: (based) loops (up to equivalence).
multiplication: concatenation



Knot $\rightsquigarrow$ fundamental group of $S^3 - K$.
Unknot $\rightsquigarrow$ $\mathbb{Z}$
Trefoil $\to$ not $\mathbb{Z}$ This is the original proof, due to Max Dehn that $\neq 0$.

We will begin by labelling knots by groups (similar to Alex poly).
First, recall....

Symmetric Groups

S_n: group of permutations of $\{1, \ldots, n\}$.
or functions (bijections) $\{1,\ldots,n\} \to \{1,\ldots,n\}$ under composition.

cycle notation: $(1\ 4\ 2)$

$1 \to 4$
$4 \to 2$
$2 \to 1$
$i \to i$ o.w.

transpositions: $(i\ j)$

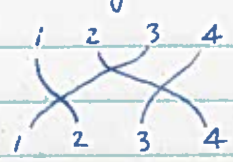
Think of S_n as follows: Start with n labelled balls
$(i\ j)$ means – switch balls in i^{th} & j^{th} spot.

Fact: Each elt of S_n is written (uniquely) as a product of disjoint cycles

Fact: Two elements of S_n are conjugate iff cycle decompositions look the same.

Thm: S_n generated by $(i\ i{+}1)$ $1 \le i \le n-1$.

Proof:



draw picture: crossings $\leftrightarrow$ transpositions.

$(1\ 2\ 4\ 3) = (2\ 3)(1\ 2)(3\ 4)$

(order of last two does not matter).

Another generating set: $(1\ 2)$, $(1\ 2 \cdots n)$

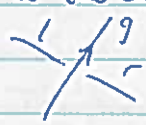
[Pf: $(i\ \ i+1) = (1\ 2 \cdots n)^{-i}(1\ 2)(1\ 2 \cdots n)^{i}$]

Labelling (again)

K = knot, with diagram

G = group.

A labelling of the diagram by G is an assignment of group elements to the arcs so that:

①  $g\,l\,g^{-1} = r$

② The labels generate G.

Fact: All labels are conjugate

Pf: The rule says each arc label is conj to next.

Fact: Orientation does not matter

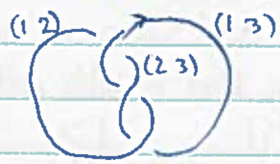
Pf: Replace each label with inverse.

Thm: Diagram doesn't matter. Any two diagrams of same knot can be labelled by same conjugacy class.

Fact: Unknot cannot be labelled by S_n $n \geq 3$.

Pf: Standard diagram labelled by one element. Groups generated by one element are abelian, but $(1\ 2) \not\leftrightarrow (2\ 3)$.

Fact: We can label trefoil:



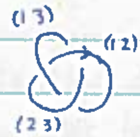
10/29.

Knots and Groups

A labelling of a knot diagram by a group G is an assignment

arcs $\to G$

subject to ①  $g l g^{-1} = r$ ② labels generate G.

We labelled trefoil with S_3: 

What about other groups we know?

$\mathbb{Z}/n\mathbb{Z}$, $\mathbb{Z}$, $\mathbb{Z}\times\mathbb{Z}$, S_n, $GL(n,\mathbb{Z})$, D_n ... ?

A new group: The Braid Group B_n

elements:  operation: concatenate identity: 

N.B. Knotting a strand  is not allowed.

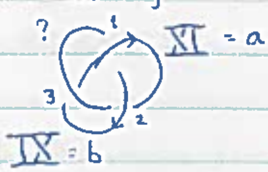
inverses: reflect.

Exercise: formalize this.

There is a map $B_n \to S_n$. This is a homomorphism.

So,  $\to (1\,2)$ etc.

we can guess a labelling of trefoil by B_3



by crossing 1,

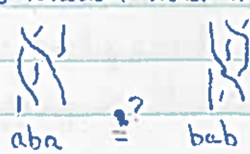
? is $a^{-1}ba$: 

To check this is a labelling, try next crossing:

Crossing 2. $bab^{-1} = a^{-1}ba$

or $aba = bab$

Does this relation hold in B_3?



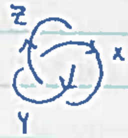
yes!

Crossing 3. $\rightsquigarrow aba = bab$ redundant!

Fact: Last crossing always redundant. (Proof?).

Finally, need to check labels generate B_3. Proof similar to proof that transpositions generate S_n (but must now remember the crossings).

Can abstract this further to find all labellings by all groups (similar to how det tells us all colorings).
Just label each arc by a letter, and list the required equalities.



$xzx^{-1} = y$
$yxy^{-1} = z \leftarrow$
$zyz^{-1} = x$

Can simplify by eliminating z:

$xyxy^{-1}x^{-1} = y$ or $xyx = yxy$
$yx\cancel{y}\,\cancel{y}yx^{-1}y^{-1} = x$ or $yxy = xyx$.

So we need a group generated by x, y and subject to $xyx = yxy$ (braid relation).
examples: B_3, S_3.
But it makes sense to talk about an abstract group with these properties...

Group Presentations

The information $\langle x, y : xyx = yxy \rangle$ is called a <u>group presentation</u>. We get a group as follows:

elements - "words" in x, y up to equivalence $w_1 x x^{-1} w_2 = w_1 w_2$
operation - concatenation. $w_1 xyx\, w_2 = w_1 yxy\, w_2$
identity - empty word.

The resulting group in this case is <u>the</u> group for trefoil.

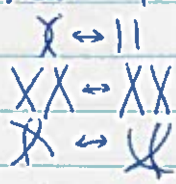
Other presentations: $\mathbb{Z}/m\mathbb{Z} = \langle t \mid t^m = 1 \rangle$
$\mathbb{Z} = \langle t \mid \rangle$
$\mathbb{Z} \times \mathbb{Z} = \langle s, t \mid st = ts \rangle$
$F_n = \langle x_1, \ldots, x_n \mid \rangle$

There is a homomorphism from F_n onto any group with n generators.

$$S_n = \langle s_1,\dots,s_{n-1} \mid \begin{array}{l} s_i^2 = 1 \\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \\ s_is_j = s_js_i \quad |i-j|>1 \end{array} \rangle$$

Prove for HW: Start with a diagram

Need to pull apart using



$s_i^2 = 1$

$s_is_j = s_js_i$

$s_is_{i+1}s_i = s_{i+1}s_is_{i+1}$.

Claim: $\langle x,y \mid xyx = yxy\rangle$ is isomorphic to B_3.
So B_3 is the universal group that labels trefoil, that is, if G is another group that labels trefoil, then there is a surjective homomorphism $B_3 \to G$.

What is more, the universal group labelling any knot is isomorphic to the fundamental group of the complement!

What is even more, we ~~can~~ will derive the Alexander polynomial from this group!

You are Here

Knot labellings $\rightsquigarrow$ the group of a knot

We will derive Δ_K from group, hence all coloring info.

What is more: group$(K) \cong$ fundamental group $(S^3 - K)$

Compute group of Fig 8 knot

Fundamental Group

X top space (surface, n-manifold, tree, etc..)

$x_0 \in X$.

$\pi_1(X, x_0) =$ fund gp of X based at x_0.

elements: loops based at x_0

up to homotopy

operation: concatenation.

More precisely:

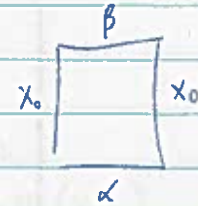
based loop = contin. map $\alpha : [0,1] \to X$ $\quad \alpha(0) = \alpha(1) = x_0$

homotopy = contin map $h : [0,1] \times [0,1] \to X$

$h|_{[0,1] \times 0} = \alpha$

$h|_{[0,1] \times 1} = \beta$

$h(\{0,1\} \times [0,1]) = x_0$

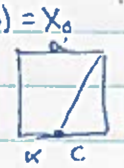


operation: $\alpha\beta(s) = \begin{cases} \alpha(2s) & 0 \le t \le 1/2 \\ \beta(2s-1) & 1/2 \le t \le 1 \end{cases}$

Check this is a group

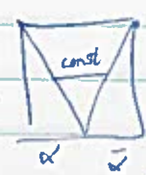
Identity = constant loop $c(s) = x_0$

check $\alpha c = c\alpha = \alpha$



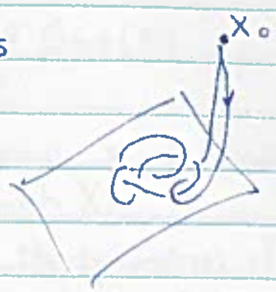
Inverses $\bar{\alpha}(s) = \alpha(1-s)$

check $\alpha\bar{\alpha} = c(s)$



Associativity $(\alpha\beta)\gamma = \alpha(\beta\gamma)$

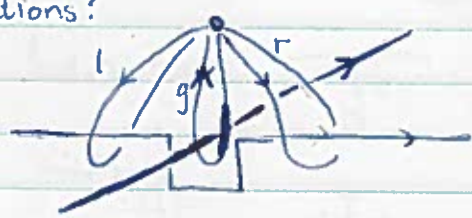
Back to knots

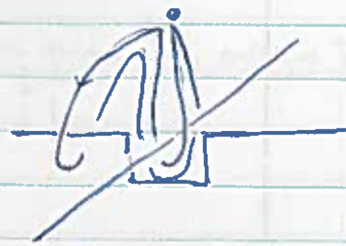


choose one loop around each arc. orient knot & orient these loops using right hand:

claim: $n-1$ of these loops generate $\pi_1(S^3 - K)$.

relations?





$$g l g^{-1} = r$$

We will show: these are all relations. (How compute π_1?)

Examples of Fundamental groups

$\pi_1(\mathbb{R}^n) = 1$ $\qquad h(s,t) = (1-t)\alpha(s)$

Fact: $x_0, x_1 \in X$. If X path connected, any path $x_0 \to x_1$ gives an isomorphism $\pi_1(X, x_0) \to \pi_1(X, x_1)$.

$\pi_1(S^n) = 1 \quad n > 1$

$\pi_1(\text{tree}) = 1$

$\pi_1(S^1) \cong \mathbb{Z}$ $\qquad$ next time

$\pi_1(\infty) \cong F_2$

$\pi_1($$) \cong \mathbb{Z}^2$

Goal: $\pi_1(S^3-K) \cong \text{Group}(K)$. 11/5

Warmup: $\pi_1(S^1) \cong \mathbb{Z}$

$\pi_1(X)$ is a group, so it should be the symmetries of something having to do with X.

What is $\mathbb{Z}$ the symmetries of?

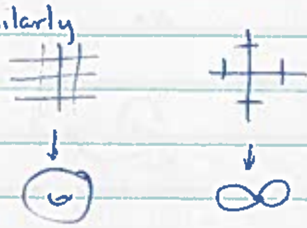
$\mathbb{Z}$: symmetries of line, preserving integers and their order.

If we identify all points equivalent under symmetries, get

And a map

paths starting at 0, ending at $\mathbb{Z}$.
$\updownarrow$
based loops

Similarly



Key idea: Given a based loop $\alpha: [0,1] \to S^1$, there is a unique path $\tilde{\alpha}: [0,1] \to \mathbb{R}$ ~~st~~ s.t. $\tilde{\alpha}(0) = 0$ and $p \circ \tilde{\alpha} = \alpha$.

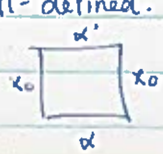
To do this, note that if you take any interval of length < 1 in S^1, the preimage under p is a collection of disjoint copies.

So, given α, cut $[0,1]$ into intervals I_i s.t. $\alpha|_{I_i}$ has length < 1. ~~∃~~ (compactness). Lift piece by piece. Done.

<u>Thm</u>: $\pi_1(S^1) \cong \mathbb{Z}$

<u>Pf</u>: We define a map $f: \pi_1(S^1) \to \mathbb{Z}$ by $\alpha \longmapsto \tilde{\alpha}(1)$.

① Well-defined. Say $\alpha \sim \alpha'$ i.e. there is a homotopy



Cut homotopy into little squares and lift the homotopy. Resulting homotopy of paths leaves endpoints at integers, hence fixed.

② Injective. $f(\alpha)=0 \Rightarrow \hat{\alpha}$ a loop $\Rightarrow \hat{\alpha}$ homotopic fixing basepoint. $p \circ$ (this homotopy) is a homotopy of α to constant map.

③ Surjective easy

④ Homomorphism easy ∎

Can now prove $\pi_1(S^3 - O) \cong \mathbb{Z}$.

Idea: If $X \approx X'$ then $\pi_1(X) \cong \pi_1(X')$

S^3 - solid torus = solid torus $\approx S^1$.

Idea for $\pi_1(S^3 - K) \cong \text{Group}(K)$.

generators for π_1

$\updownarrow$ $\searrow$

walls $\longleftrightarrow$ arcs

relations for π_1

$\updownarrow$ $\searrow$

intersections of walls $\longleftrightarrow$ $olo^{-1} = r$

Tying up some loose ends 11/10.

$\pi_1(S^3 - O) \cong \mathbb{Z}$

Fact: If X is a deformation of Y then $\pi_1(X) \cong \pi_1(Y)$

examples: $\pi_1(\text{tree}) = 1$, $\pi_1(\text{surface})$, letter of alphabet, annulus, pants, $S_{1,1}$.

Fact: $S^3 = S_g \cup S_g$ for any g. In particular, $g=1$.

$\pi_1(S^3 - O) \cong \pi_1(S^3 - \overset{\text{solid}}{T^2}) \cong \pi_1(\text{solid torus}) \cong \pi_1(S^1) \cong \mathbb{Z}$.

Does π_1 determine the space, up to deformation?

No $\pi_1(S^2) \cong \pi_1(\cdot)$ but no deformation.

Poincaré Conjecture: Every simply connected, closed, orientable 3-man is S^3.

Does $\pi_1(S^3 - \text{knot})$ determine K? No: Left, right trefoils.

Does $S^3 - K$ determine K? Of course! Sure? . . .

Not true for links!

Gordon Luecke Thm 1989

A knot is determined by its exterior

Obtaining $\Delta_K(t)$ from knot group

Fox derivative $\mathbb{Z}[G] \to \mathbb{Z}[G]$ group ring $G = F_n$.

$\frac{d}{dx_i}(x_j) = \delta_{ij}$

$\frac{d}{dx_i}(1) = 0$

$\frac{d}{dx_i} wz = \frac{d}{dx_i} w + w \frac{d}{dx_i} z$

Easy facts: $d/dx_i\,(x_i^{-1}) = -x^{-1}$

$d/dx_i\,(w) = 0$ if $w \in \langle x_j \mid j \neq i \rangle$.

Procedure: Start with a knot group presentation:

$k+1$ gens, k relators.

$\rightsquigarrow$ Jacobian matrix

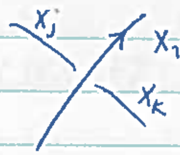
Take determinant, plug in t for each x_i

Tada!

Check for trefoil: $\langle x,y : xyx = yxy \rangle$

fig 8 : $\langle x,y : x^{-1}yxy^{-1}xy = yx^{-1}yx \rangle$

Start with canonical presentation:



$x_i x_j x_i^{-1} = x_k$

Take 3 derivatives, plug in t, get $1-t, -1, t$.

Amazing fact: if you change the presentation, it still works! Cf. Fig 8 example above.

The next invariant — Linking number 11/12.

(we will derive Alex poly from this, too!).

Crossing index 

+1 -1

J, K knots, $Lk(J,K) = \frac{1}{2}\Sigma$ crossing indices at all crossings where J, K meet.

Fact: $Lk(J,K)$ is an invariant of a 2-component link. (Reid. moves).

What is linking number of Whitehead Link? Answer: Zero!

How to get positive linking number? Hopf Link

Arbitrary linking number: $(2,n)$ torus link, n even.

Fact: $Lk(J,K)$ is an integer.

Pf: Checkerboard ⇒ even # of crossings.

Fact: $Lk(J,K)$ = sum of crossing indices where J over K

Pf: Consider the difference of the sums where J over K and where K over J. This difference is invariant under Reid moves & change of crossings, thus it is same as unlink, so it is zero.

Seifert Matrix

If C is a simple closed curve in an orientable surface embedded in $\mathbb{R}^3$, then we can form the pushoff C^*. Just, well, push off a tiny bit in some prechosen normal dir.

Q. Can we get $Lk(C, C^*) \neq 0$?

yes:  or 

~~Seifert Matrix~~ is

$\cancel{\{m_{ij}\}}$ $\cancel{m_{ij}} =$

Given a knot K, find a Seifert surface. ~~Choose~~ Write surface as a disk with bands. If genus is g, have $2g$ bands: $\chi(S_{g,1}) = 1-2g$ $\chi(\text{Disk} \cup b \text{ bands}) = 1-b$.

Let $c_1, \ldots, c_{2g}$ be core curves for bands:

The Seifert Matrix for K is $M = (m_{ij})$

$$m_{ij} = Lk(c_i, c_j^*).$$

Example: Trefoil knot

Check this is trefoil!

compute M.

Claim: $\Delta_K(t) \doteq \det(V - tV^t)$

Can check in this example

What about mirror image?

Will use Seifert Matrix to show left- & right-handed trefoils are different.

11/17

Cmt:  $\neq$ 

<u>Theorem</u>: If V is a Seifert matrix for K, then
$$\Delta_K(t) = \det(V - tV^t)$$

Proof? Will at least show this RHS is invariant.

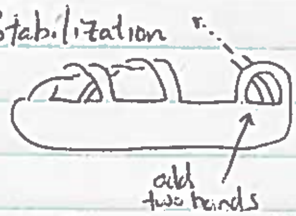
<u>Cor</u>: $\Delta_K(t) = t^{\pm ?}\,\Delta_K(t^{-1})$ some ?

<u>Pf</u>: $\Delta_K(t) = \det(V - tV^t) = \det((V - tV^t)^t)$
$= \det(V^t - tV) = \det(tV - V^t)$ (even dim matrix)
$= \det(t(V - t^{-1}V^t))$
$= t^{2g}\,\Delta_K(t^{-1})$.

<u>Theorem</u>: $\det(V - tV^t)$ is a knot invariant (up to $\pm t^k$).

<u>Proof sketch</u>: Will show all Seifert surfaces differ by two types of moves, neither of which affect determinant.

Move 1: Stabilization



extra band allowed to do anything.

New Seifert matrix
$$\begin{pmatrix} V & \begin{matrix} * & 0 \\ * & 0 \\ * & 0 \end{matrix} \\ \begin{matrix} * & * & * \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} * & 1 \\ 0 & 0 \end{matrix} \end{pmatrix}$$

find $V' - tV'^t$, expand out, det ~~changes~~ mult. by t.

Move 2: Band slide.

$\leftarrow$ just new curve drawn.

$a, b \longrightarrow ab, b$.

Corresponds to simultaneous row/col operations.

$\Rightarrow$ det unchanged. $\blacksquare$

<u>Signature</u>

Fact: Every symmetric matrix is congruent to diagonal, that is, if M symm, $\exists$ orthogonal P s.t. PMP^t is diagonal D.

Def: In resulting D, # pos entries $-$ # neg entries is the <u>signature</u> $\sigma(M)$.

If K is a knot, its signature is $\sigma(K) = \sigma(V+V^t)$
where V is Seifert matrix for K.

To find D, do simultaneous row/col ops on $V+V^t$.

examples: Left hand trefoil (See Murasugi)

$$\begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \Rightarrow \sigma = -2$$

Right hand trefoil $\Rightarrow \sigma = +2$

So left & right trefoils are different, if we show:

Thm: σ is a knot invariant.

Pf: Check what happens under band slides & stabilization.

Use Sylvester's Theorem: if B symm & $B = MAM^t$ where M invertible, then $\sigma(A) = \sigma(B)$.

11/20

Today: a new polynomial!

Jones Polynomial 1984 modern era.

There are no known examples of nontrivial knots with trivial JP!

(There are "mutant" knots with same JP).

first Kauffman Bracket Polynomial. $\langle L \rangle$

axiomatic — what rules should a knot polynomial satisfy?

Rule 1: $\langle O \rangle = 1$

Rule 2: $\langle \times \rangle = A\langle)(\rangle + B\langle \asymp \rangle$

Rule 3: $\langle L \cup O \rangle = C\langle L \rangle$

Is this an invariant?

Type II move

$$\langle \quad \rangle = \cdots = (A^2+ABC+B^2)\langle \asymp \rangle + BA\langle)(\rangle = \langle)(\rangle$$

$$\Rightarrow B = A^{-1}, \quad C = -(A^2+A^{-2}).$$

$\rightsquigarrow$ refined rules.

Type III move

$$\langle \quad \rangle = A\langle \quad \rangle + A^{-1}\langle \quad \rangle$$ use type 2 twice

$$= A\langle \quad \rangle + A^{-1}\langle \quad \rangle$$

$$= \langle \quad \rangle \quad \checkmark$$

Computations

$\langle \text{trivial } n\text{-component link} \rangle = (-A^2-A^{-2})^{n-1}$

$\langle \text{Hopf} \rangle = -A^4 - A^{-4}$

$\langle \quad \rangle = ?$

Type I move

$$\langle \quad \rangle = A\langle \quad \rangle + A^{-1}\langle \quad \rangle$$

$$= A\langle \frown \rangle + (A^{-1})(-A^2-A^{-2})\langle \frown \rangle$$

$$= -A^{-3}\langle \frown \rangle$$

So $\langle\ \rangle$ not an invariant!

Fix it with another noninvariant. (unless $A=-1$)

Writhe

+1 $\quad$ −1

$\omega(L) = \Sigma$ indices

like self-linking number

ω is not an invariant, but, like $\langle\ \rangle$, it is invariant under moves II & III.

Can now fix $\langle\ \rangle$.

$$\boxed{X(L) = (-A^3)^{-\omega(L)} \langle L \rangle}$$

Check this is invariant (better derive it).

Does orientation matter?

Jones Polynomial $\quad A = t^{-1/4}$.

11/24.

Bracket Polynomial

$$\langle \bigcirc \rangle = 1$$
$$\langle L \cup \bigcirc \rangle = (-A^2 - A^{-2})\langle L \rangle$$
$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

examples: $\langle \infty \rangle = -A^3$ observation: switching all crossings negates all exponents.

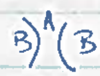
$\langle \infty \rangle = -A^{-3}$

$\langle \infty\!\circ \rangle = A^6$ $\langle \infty\!\circ \rangle = A^{-6}$

$\langle @ \rangle = -A^4 - A^{-4} = \langle @ \rangle$

$\langle \cdot \rangle = A^7 - A^3 - A^{-5}$ $\langle \cdot \rangle = A^{-7} - A^{-3} - A^5$

In above examples, calculation is inductive, using previous examples. Can give closed formulation. At each crossing label regions: 

Change crossing by A-split:  (open "channel" between A regions).

B-split: 

A state S for a diagram is a choice of split at each crossing. Let $a(S) = \#$ A-splits $|S| = \#$ components.

$b(S) = \#$ B-splits

Claim: $\langle L \rangle = \sum_S A^{a(S)} A^{-b(S)} (-A^2 - A^{-2})^{|S|-1}$

In particular, $\langle L \rangle$ exists, i.e. it is well-defined.

Kauffman Polynomial

$X(L) = (-A^3)^{-w(L)} \langle |L| \rangle$ Notes: w, $\langle\ \rangle$ don't depend on orientation of diagram or type II, III moves.

Check that X invariant under type I:

$\langle \ell \rangle = -A^{-3} \langle \frown \rangle$ $w(\ell) = w(\frown) - 1$

so $X(\ell) = (-A^{-3})(-A^3) X(\frown) = X(\frown)$ ✓

$\Rightarrow$ Left & right trefoils are distinct!

Next: Use X & $\langle\ \rangle$ to give easy proofs of long standing conjectures.

Alternating Knots

A diagram is <u>alternating</u> if, following the knot in some <u>direction</u>, the crossings go over, under, over, under, ... eg 

A knot is <u>alternating</u> if it has an alternating diagram.

A <u>reduced</u> alternating diagram is one without isthmuses:

Crossing number

$c(K)$ = min number of crossings in a diagram.

$c(\bigcirc) = 0$

$c(\;) = 3$ (proof: we listed all diagrams with ≤ 2 crossings).

Hard to compute! <u>Conj</u>: $c(K_1 \# K_2) = c(K_1) + c(K_2)$.

We will prove the following two conjectures from 1800s:

<u>Conj 1</u>: Two reduced alternating diagrams of the same knot have same number of crossings.

<u>Conj 2</u>. A reduced alternating diagram for a knot realizes the crossing number.

Also:

<u>Tait Flyping Conjecture</u>: Any two reduced alternating diagrams for the same knot differ by a sequence of "Flypes"

12/1

Alternating knots. Note all primes with ≤ 7 crossings are alt.

Reduced diagrams: no isthmus 

Bracket polynomial $\sum_S A^{a(S)} A^{-b(S)} (-A^2 - A^{-2})^{|S|-1}$

note: span$\langle \cdot \rangle$ is an invariant.

Thm 1: Any two red. alt. diagrams of a knot K have same number of crossings. In particular, this number is span$\langle K \rangle/4$

Thm 2: A nonalternating diagram of an alt knot K has more than span$\langle K\rangle$ crossings.

So: The crossing number of an alt knot K is ② the number of crossings of any ① reduced alt diagram. (Do any random example)

Lemma 1: Given a diagram with n crossings, the number of regions is $n+2$.

Pf: Euler char: $E = \frac{4n}{2}$ $V - E + F = n - 2n + F = 2 \Rightarrow F = n+2$.

Proof that $\chi(S^2) = 2$. Remove one triangle at a time.

Lemma 2: Given a black & white checkerboarding of an alternating diagram, as below an A-split always opens a channel between black regions, vv.

Pf: Inspection 

random example

Proof of Thm 1: Let D be a reduced alt. diagram.

What is highest degree of $\langle D \rangle$?

Guess: A-state $n + 2(|S_A| - 1) = n + 2(W-1)$

A-state lets black waters flood, left with W white islands, where W = # white regions. (Lemma 1)

This is actually maximal since, any time you switch an A-split to a B-split, $a(S) - b(S)$ goes down by 2, and $|S|$ changes by at most 1. But we need to do better — a priori if you have two terms with maximal degree, they might cancel.

Observe: ~~Starting fr~~ If changing an A-~~state~~ split to a B-~~state~~ split connects two islands, then $|S|$ goes down by one. 

If it connects an island to itself, $|S|$ goes up by one. 

Thus starting from A-state, the hypothesis of reduced (no isthmuses), changing one A-~~state~~ split to a B-split connects two islands, and so $|S|$ goes down by one.

To summarize: If M is the degree coming from S_A (A-state) then the degree of the term from any state S where one split is changed to a B-split is at most $M-2$. Changing more A-splits to B-splits makes $a(S)-b(S)$ go down by 2 and $2(|S|-1)$ changes by at most 2, so all other terms have degree at most $M-2$.

So $n+2(W-1)$ is the highest degree in $\langle D\rangle$.

By symmetry, lowest degree is $2(1-B)-n$, so

$$\begin{aligned} \mathrm{span}\langle D\rangle &= n+2(W-1)-(2(1-B)-n) \\ &= 2n+2(W+B-2) \qquad \text{apply Lemma 2} \\ &= 2n+2n = 4n. \end{aligned}$$

∎

For a state S, let S_d denote dual state (switch all splits).

Fact 1: For any diagram D, $|S|+|S_d| \leq R$ (= # regions) (see Justin Roberts knotes (induction))

Lemma 1 ⇒ equality in alt. case.

Prop 1: For any reduced diagram D with n crossings, $\mathrm{span}\langle D\rangle \leq 4n$.

(same proof, but use Fact 1).

Fact 2: For a nonalternating diagram, $|S|+|S_d| < R$

Pf: Consider 

~~Prop 2~~ Again, same proof implies our Thm 2 above.

So reduced alternating diagrams realize crossing number, and nonalternating diagrams do not. In our random example, crossing number is 7!

Mirror Images

Suppose $K \approx K^*$ (mirror image) K^* obtained by switching all crossings.

Then $X(K) = X(K^*) = X(K)(A^{-1})$

$\Rightarrow$ max deg $X(K)$ = − min deg $X(K)$.

If further K is alternating; we know max/min degrees:

$$-3\omega(K) + n + 2(W-1) = 3\omega(K) + n + 2(B-1)$$

$$\Rightarrow \quad 3\omega(K) = W - B.$$

In above example, $W - B = 1$, so $K \not\approx K^*$!

12/8

Jones Polynomial

Next goal: rework Jones so it is defined by 3 axioms (without any kind of writhe fix). Will lead to a wealth of other invariants.

Start by writing third bracket rule in two ways.

$$\langle \times \rangle = A \langle)(\rangle + A^{-1} \langle \asymp \rangle$$

$$\langle \times \rangle = A^{-1} \langle)(\rangle + A \langle \asymp \rangle$$

Multiply first by A, second by A^{-1}, and add:

$$A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle)(\rangle$$

Want to relate this to writhe (hence X poly.), so we introduce orientations:

$$A \langle \times \rangle - A^{-1} \langle \times \rangle = (A^2 - A^{-2}) \langle)(\rangle$$

Write this as:

$$A \langle L^+ \rangle - A^{-1} \langle L^- \rangle = (A^2 - A^{-2}) \langle L^0 \rangle$$

Note $w(L^{\pm}) = w(L^0) \pm 1$. So, switching from $\langle\ \rangle$ to X:

$$A(-A^3) X(L^+) - A^{-1}(-A)^{-3} X(L^-) = (A^2 - A^{-2}) X(L^0)$$

$$-A^4 X(L^+) + A^{-4} X(L^-) = (A^2 - A^{-2}) X(L^0). \quad (*)$$

"Define" a polynomial by: (oriented links)

Rule 1 (Invariance) $V(L) = V(L')$ if $L \approx L'$.

Rule 2 (Normalization) $V(O) = 1$.

Rule 3 (Exchange rule) $A^4 V(L^+) - A^{-4} V(L^-) = (A^{-2} - A^2) V(L^0)$.

This is the axiomatic definition of the Jones polynomial.

Existance: X satisfies all 3 rules.

Uniqueness: Order all oriented links by lexicographic order on (C_L, U_L) where C_L is crossing number and U_L is unknotting number. Can always apply rule 3 to see that $V(L)$ is defined in terms of V of other links that come earlier in the ordering. Done by induction.

To get the real Jones polynomial, substitute $q^{-1/4}$ for A.
This is not a new polynomial, but it is nicer in that, for knots, V is a polynomial in $A^{\pm 4}$, or, $q^{\pm 1}$.

The Jones polynomial distinguishes all knots with ≤ 13 crossings. (It was thought that two knots had same J.P., but it turned out they were the same knot — the "Perko pair").

Lemma 1: L = unlink on m components. $V(L) = (-q^{-1/2} - q^{1/2})^{m-1}$
Lemma 2: L^+, L^- have same number of components mod 2, L^0 has different parity.

Theorem: If # of components of L is odd, $V(L) \in \mathbb{Z}[q, q^{-1}]$.
If even, $V(L) \in \mathbb{Z}[q^{1/2}, q^{-1/2}]$.

Proof: Lemma 1 $\Rightarrow$ Thm holds for trivial links:

$$(-q^{-1/2} - q^{1/2})^{m-1} = q^{(m-1)/2}(-q^{-1}-1)^{m-1}$$

By exchange relation, V computed from V of trivial links. By similar induction as in uniqueness of V suffices to check that if Thm holds for two of three links in exchange relation, then it holds for the third. This follows from Lemma 2. $\boxtimes$

Compute X(trefoil) using axioms $-A^{16} + A^{12} + A^{4}$.

12/10

Polynomials – what is the deal?

One advantage of axiomatic definition of $V(L)$ is that it leads you to other invariants. Just replace coefficients with new ones:

$$\square\, P(L^+) + \square\, P(L^-) = \square\, P(L^0).$$

Jones	q	$-q^{-1}$	$1/\sqrt{q} - \sqrt{q}$	
Conway	1	-1	z	Recovers Alexander
Alexander	1	-1	$\sqrt{t} - 1/\sqrt{t}$	
HOMFLY	x	$-t$	1	Recovers Jones.
Jones 2-var	$1/\sqrt{\lambda}\sqrt{q}$	$-\sqrt{\lambda}\sqrt{q}$	$\sqrt{q} - 1/\sqrt{q}$	Recovers all others!

Hard part: showing these are invariants! For Jones, we did it via bracket. If we believe this Conway-Alex poly is same as old one, then it is an invariant, but we haven't shown that.

We'll show Conway poly $\nabla(L)$ is an invariant. Idea: show constant term is an invariant, linear term is an invariant, etc. (by induction). Each term actually <u>means</u> something.

Examples of Conway Polynomial

$\nabla(L) = 0$ whenever L is a <u>split link</u> 

$\nabla(\text{Hopf Link}) = z$

<u>Trefoil</u> If L^+ is right trefoil, L^- is unknot, L^0 is pos. Hopf Link.

$\Rightarrow \nabla(\text{trefoil}) = 1 + z^2$

Again, using inductive procedure with lexicographic order, as in case of Jones poly.

<u>Prop</u>: If ∇, ∇' are two functions satisfying the three axioms, then $\nabla = \nabla'$

<u>Pf</u>: Induction again.

N.B. Uniqueness works for any choice of coefficients. Existence is hard part.

Lemma: If K^+ has n components, then K^- has n components, and K^0 has: ① $n-1$ components if two components meet at the crossing in question
② $n+1$ components otherwise.

Well-definedness

Step 0: a_0 is an invariant.

Prop: $a_0(K) = 1$ if K is a knot (link with one component)
$= 0$ o.w.

Pf: Look at exchange relation

$$\nabla(L^+) - \nabla(L^-) = z\nabla(L^0)$$

const. term $= 0$.

So $\nabla(L^+) - \nabla(L^-)$ have same const. term. In other words, can switch crossings at will without changing the constant term. So switch until get unknot ☒

Step 1: a_1 is an invariant

Prop: $a_1(L) = Lk(L)$ if L is a link with two components
$= 0$ o.w.

Pf: Similar argument: First do case of knot.

$$\nabla(K^+) - \nabla(K^-) = z\nabla(K^0)$$

lin term is const. term of $\nabla(K^0) = 0$.

So difference of lin. terms is zero. Switch crossings until unknot.

Now assume L is 2-component link, crossing at pt. in question.

$$\nabla(K^+) - \nabla(K^-) = z\nabla(K^0)$$

So difference in linear terms is 1. But same is true for linking numbers. So statement true iff true for split ~~un~~unlink (can change crossings at will). Done.

Other cases are exercise.

a_2 is harder, but idea is to continue by induction to get rest.

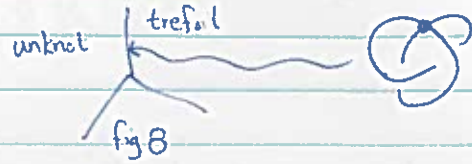
12/10

What is next?? Vassiliev Invariants 1990s

Idea: Instead of looking at one knot, study space of all knots.

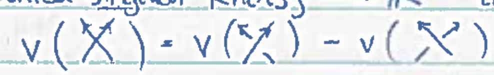
Not an interesting space: unknot trefoil fig 8

Unless, we allow these sets to interact ⇝ singular knots:



A Vassiliev Invariant is a function

v: {oriented singular knots} $\longrightarrow \mathbb{R}$ that satisfies



(*)

It turns out: Coefficients of all of our polynomials are Vassiliev invariants.

So Vassiliev invariants subsume previous invariants.

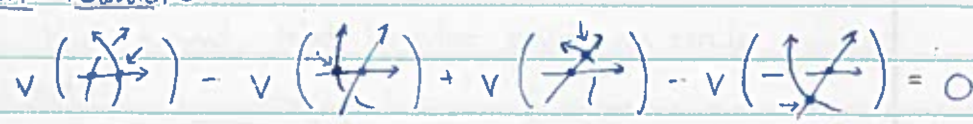
Conjecture: Vassiliev invariants distinguish all knots.

One-term relation

since 

More generally: 

Four-term relation



To check, apply (*) 4 times, at indicated singular points.

The 8 terms cancel in pairs.

It turns out: all relations are consequences of these.

Vassiliev invariants form a vector space over $\mathbb{R}$.
Will look at increasing union of fin. dim. subspaces.

Def: A Vassiliev invariant is of order no greater than n if it vanishes on any knot with $> n$ sing. pts.

Let $V_n = \{v : \text{ord}(v) \leq n\}$

$$V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots$$

$V_0 \cong \mathbb{R}$

(*) $\Rightarrow$ Can switch crossings without changing value.
So $v \in V_0$ determined by $v(O)$.

$V_1 \cong \mathbb{R}$ (still!)

(*) $\Rightarrow$ Can switch crossings on any singular knot. So for any K with one sing pt, $v(K) = v(\infty) = 0$.
$\Rightarrow V_1 \subseteq V_0 \Rightarrow V_1 \cong \mathbb{R}$.

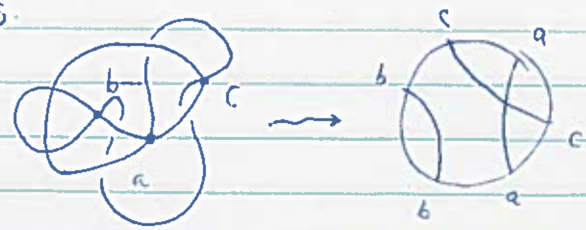
We have seen:

Lemma: The value of a Vassiliev invariant of order n on a knot with n singular pts does not vary with crossing changes.

How to describe a knot where crossings can change but singular points cannot?

Gauss diagrams

Follow knot around, label singular points on circle, connect by chords.



Gauss diagrams of order 0 : $\bigcirc$

1 :  trivial by one-term relation

2 :  , $\otimes$

trivial by 1-term relation.

So $v_2 \in V_2$ determined by $v(\bigcirc)$, $v(\otimes)$.

For example, set $v(\bigcirc) = 0$

$v(\otimes) = 1$

Compute for trefoil:

v() = v() + v() → 0

= v() + v() → 0

$= v(\otimes) = 1$. So  $\neq 0$.

Figure 8 knot:

v() = v() + v() → 0

= − v() + v() → 0

$= -v(\otimes) = -1$.