Scores: 1 2 3 4 5 E

Name key

Section HP ___

Mathematics 1553

Midterm 1

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1. Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and let T_A be the associated linear transformation.

What is the domain of T_A? $\boxed{\mathbb{R}^4}$

A is 3×4 matrix, so it transform $\mathbb{R}^4$ to $\mathbb{R}^3$

Is T_A one-to-one? $\boxed{\text{No}}$

Because one-to-one requires A to have pivot in every column, but A only has 2 pivots.

Is T_A onto? $\boxed{\text{No}}$

Because onto requires A to have pivot in every row, but A only has 2 pivots

What is the dimension of the set of solutions to $Ax = 0$ (that is, how many free variables)?

2 Free variable.

The matrix A has 4 columns, which means there are 4 variables in the system. It also have 2 pivots, which means 2 of the variable is not free.

So # free variable = # colns − # pivots = 2.

Does $Ax = b$ have a solution for every b in $\mathbb{R}^3$?

$\boxed{\text{No}}$

There are only two pivot columns in A, therefore there is only 2 L.I. columns. 2 Linearly independant colums cannot span $\mathbb{R}^3$.

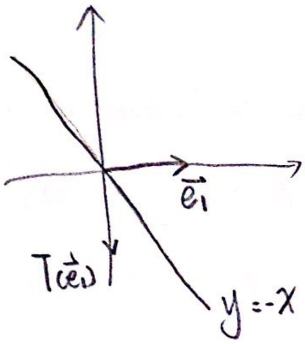
So $A\vec{x} = \vec{b}$ connot have solution to any $\vec{b}$ in $\mathbb{R}^3$.

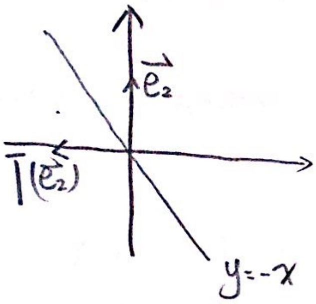
What is the span of the columns of A? $\boxed{\text{x-y plane}}$

The two L.I. columns are $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$. Therefore the span of columns of A is just the span of $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\\0\end{pmatrix}$, which is the xy-plane

2. *(a)* Find a matrix so that the associated linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ is reflection about the line $y = -x$.

$T(\vec{e_1}) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ $T(\vec{e_2}) = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$





$A = \left(T(\vec{e_1}) \ \ T(\vec{e_2}) \right)$

$= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

Find three vectors u, v, and w in $\mathbb{R}^3$ so that u and v are linearly independent but u, v, and w are linearly dependent.

eg. $\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{w} = \vec{u} + \vec{v} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

If a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto, what can you say about m and n?

(a) $m < n$

(b) $m \leq n$

(c) $m = n$

(d) $m \geq n$

(e) $m > n$

(f) I cannot say any of these definitively

onto means A has pivot in every row.

So # pivots = m

But we have # pivots $\leq$ # colns

and # pivots $\leq$ # rows

(b) Consider the linear system

$$x + y = 1$$
$$x + ky = b$$

Find a k and a b so that the system is inconsistent.

Form augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & k & b \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & k-1 & b-1 \end{array}\right)$$

if inconsistent, then there is a pivot in the last column of the augmented matrix.

So: $\begin{cases} k-1=0 \\ b-1\neq 0 \end{cases}$ $\begin{cases} k=1 \\ b\neq 1 \end{cases}$ I pick $\boxed{\begin{cases} k=1 \\ b=0 \end{cases}}$

Find a k and a b so that the system has infinitely many solutions.

if the system have infinitely many solution, it has a zero row at the bottom:

$\begin{cases} k-1=0 \\ b-1=0 \end{cases} \Rightarrow \begin{cases} k=1 \\ b=1 \end{cases}$

Find a k and a b so that the system has exactly one solution.

if the system just have one solution, then

$\begin{cases} k-1\neq 0 \\ (b-1) \text{ is not constrain} \end{cases}$

So $\begin{cases} k\neq 1 \\ b \text{ is free} \end{cases}$ I pick $\begin{cases} k=0 \\ b=0 \end{cases}$

3. Complete ONE of the following two problems. Circle the letter of the one you chose.

(a) Determine all values of x so that the vectors $(1, 1, x)$, $(1, x, 1)$, and $(x, 1, 1)$ are linearly dependent. Explain your answer.

form a matrix A:

$$\begin{pmatrix} 1 & 1 & x \\ 1 & x & 1 \\ x & 1 & 1 \end{pmatrix} \xrightarrow[R_3 - xR_1 - R_3]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & x \\ 0 & x-1 & 1-x \\ 0 & x-1 & x^2-1 \end{pmatrix} \xrightarrow{R_3 \to R_2 - R_3} \begin{pmatrix} 1 & 1 & x \\ 0 & x-1 & 1-x \\ 0 & 0 & x^2+x-2 \end{pmatrix}$$

if the three vectors are L.D.

then $x^2+x-2=0$ or $x-1=0$

$\hookrightarrow$ $(x+2)(x-1)=0$ or $x=1$

So $x=-2$ or $x=1$

(b) For which values of h is $(1, 2, h)$ in the span of $(1, 1, 2)$ and $(1, 1, 3)$? Explain your answer.

Assume $\begin{pmatrix} 1 \\ 2 \\ h \end{pmatrix} = a\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} + b\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

then we have:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ h \end{pmatrix}$$

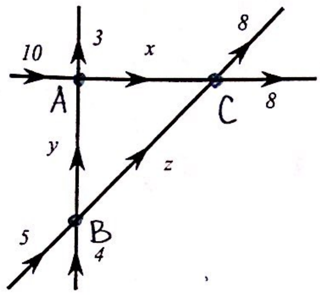
form an augmented matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 2 & 3 & h \end{pmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & h-2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & h-2 \\ 0 & 0 & 1 \end{pmatrix}$$

Since there is always a pivot in the last column of the argumented matrix, the system is always inconsistent.

So there is no h that can make (1, 2, h) in the span.

4. The following diagram indicates traffic flow in one part of town:



Write a system of linear equations in x, y, and z describing the traffic flow around the triangle.

The # car into the point should equal to the # car out.

$$\begin{aligned} A&: 10+y=3+x \\ B&: 5+4=y+z \\ C&: x+z=8+8 \end{aligned} \quad \Rightarrow \quad \begin{cases} x-y=7 \\ y+z=9 \\ x+z=16 \end{cases}$$

Turn the system of linear equations from the first part into an augmented matrix A.

$$\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 16 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 16 \end{array}\right)$$

Copy your matrix A from the previous page. Find the reduced row echelon form of A.

$$\left(\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 0 & 1 & 1 & 9 \\ 1 & 0 & 1 & 16 \end{array}\right) \xrightarrow{R_3 \to R_3 - R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 7 \\ 0 & 1 & 1 & 9 \\ 0 & 1 & 1 & 9 \end{array}\right) \xrightarrow[R_3 \to R_2 - R_3]{R_1 \to R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 16 \\ 0 & 1 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Write down the set of solutions of the original system of linear equations.

$$\begin{cases} x = 16 - z \\ y = 9 - z \\ z \text{ is free} \end{cases}$$

Write the set of solutions to the original system of linear equations in parametric form.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 16 \\ 9 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}$$

What is the maximum amount of traffic on the road labeled z?

$$\begin{cases} x = 16 - z \geq 0 \\ y = 9 - z \geq 0 \end{cases} \Rightarrow z \leq 9$$

So $\max(z) = 9$

5. Consider the matrix

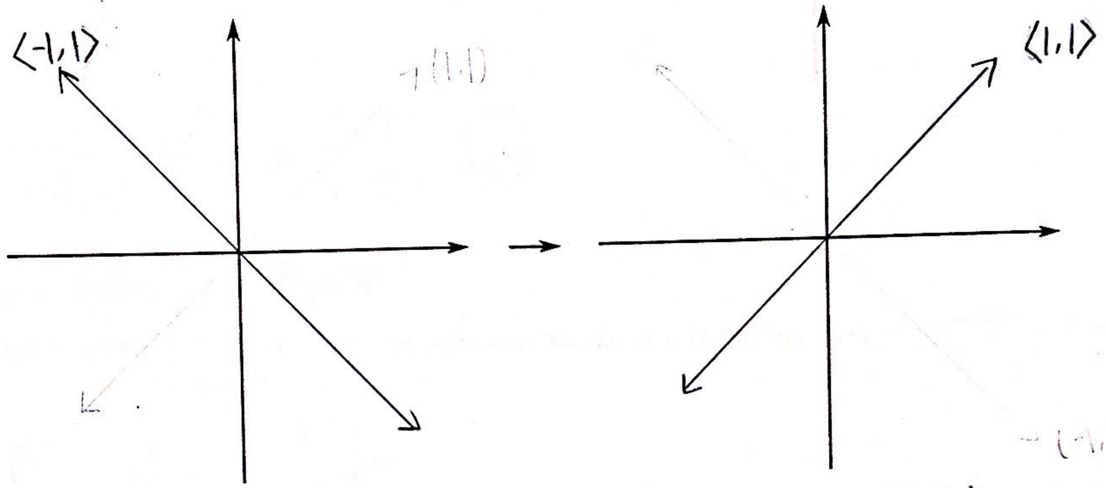
$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Let T_A be the associated linear transformation.

What is $T_A(100, 1)$?

$$T_A\begin{pmatrix}100\\1\end{pmatrix} = \begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}100\\1\end{pmatrix} = \begin{pmatrix}101\\101\end{pmatrix}$$

On the right-hand side draw the range of T_A. On the left-hand side draw the set of points v in the domain with $T_A(v) = 0$.



$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

$$\begin{pmatrix}x+y\\x+y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix} \Rightarrow x+y=0 \Rightarrow y=-x$$

$$\begin{pmatrix}1&1\\1&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x+y\\x+y\end{pmatrix} = (x+y)\begin{pmatrix}1\\1\end{pmatrix}$$

The picture on the left shows:

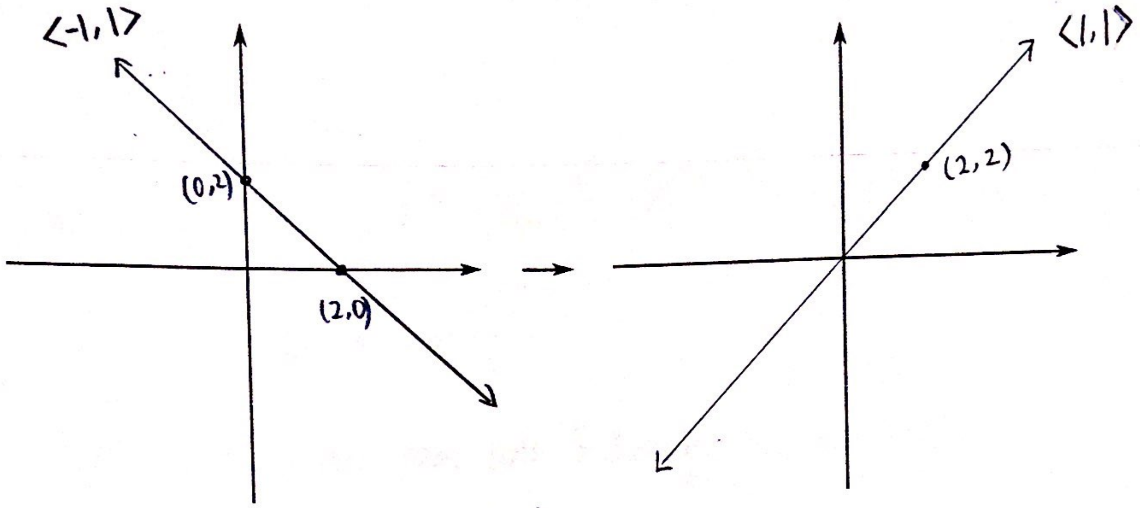
(a) The solutions to $Ax = 0$

(b) The product of A with the 0 vector

(c) The solutions to $Ax = b$ for all b

(d) None of the above

On the right-hand side draw the range of T_A again. Choose any nonzero point a in the image of T_A and label it with its (x, y)–coordinates. On the left-hand side draw the set of points v in the domain with $T_A(v) = a$.



$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$\hookrightarrow x+y=2 \Rightarrow y=2-x$

Is there a vector b in $\mathbb{R}^2$ so that the solutions to $Ax = b$ is the line $x + y = 6$? If so, find such a b; if not, explain why not.

$$\vec{b} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x+y \end{pmatrix}$$

Since $x+y=6$

$$\vec{b} = \begin{pmatrix} x+y \\ x+y \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}$$

Is there a vector b in $\mathbb{R}^2$ so that the solutions to $Ax = b$ is the line $x - y = 6$? If so, find such a b; if not, explain why not.

$\boxed{\text{No}}$

because the solution to $A\vec{x} = \vec{b}$ must be parallel to the homogenous solution which is $x+y=0$. Apparently $x-y=6$ is not parallel to $x+y=0$.

Extra credit. Find all geometric sequences that are Fibonacci sequences (meaning, each term is the sum of the two previous terms). Use this to describe the set of all Fibonacci sequences.

Assume the sequence is

$a_n = a_0 r^{n-1}$

$a_n + a_{n+1} = a_{n+2} \Rightarrow \cancel{a_0} r^{n-1} + \cancel{a_0} r^n = \cancel{a_0} r^{n+1} \Rightarrow r^2 = r + 1$

So $r^2 - r - 1 = 0 \Rightarrow r = \frac{1 \pm \sqrt{5}}{2}$

So for $\varphi = \frac{1+\sqrt{5}}{2}$, we have one Fibonacci sequence of

$a_n = a_0 \varphi^{n-1}$, where a_0 is free

for $\Phi = \frac{1-\sqrt{5}}{2}$, we have another Fibonacci sequence:

$b_n = b_0 \Phi^{n-1}$, where b_0 is free

So all the other Fibonacci (F_n) sequences are just the linear combination of a_n and b_n.

$F_n = a_0 \varphi^{n-1} + b_0 \Phi^{n-1}$