Name Prof M

Section HP ___

Mathematics 1553

Midterm 2

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1. Consider the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Find a basis for the column space of A.

$$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

What is the dimension of the column space of A?

1

2. Consider again the matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Find a basis for the null space of A.

$A \rightsquigarrow x_2 = 0$, x_1 & x_3 free

$\rightsquigarrow x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\rightsquigarrow \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$

What is the dimension of the null space of A?

2

3. Let

$$b_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \text{ and } b_2 = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

and $B = \{b_1, b_2\}$. Explain why B is a basis for $\mathbb{R}^2$.

- $\{b_1, b_2\}$ is linearly ind. since both b_1 & b_2 are nonzero and b_2 is not a multiple of b_1
- any two linearly indep. vectors form a basis for $\mathbb{R}^2$, since $\mathbb{R}^2$ is 2-dimensional.

What is $[e_1]_B$? In other words, what are the B-coordinates of $e_1 = (1, 0)$?

$$\left(\begin{array}{cc|c} 1 & -2 & 1 \\ -2 & 5 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

$(5, 2)$

4. Consider the set of vectors

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

with $xyz = 0$. Do these vectors form a subspace of $\mathbb{R}^3$? Explain your answer.

No. $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are in the set but their sum is not.

Suppose that A is a 5×2 matrix and that the image of the associated linear transformation is a 2–dimensional plane. Describe the set of solutions to $Ax = 0$. Explain your answer.

$$\text{rk}\, A + \dim \text{Nul}\, A = \#\text{ cols of } A$$

$$2 + \dim \text{Nul}\, A = 2$$

$$\Rightarrow \dim \text{Nul}\, A = 0$$

$$\Rightarrow \text{Nul}\, A = \{0\}$$

5. Find an LU factorization of the following matrix:

$$A = \begin{pmatrix} 5 & -1 & 2 & 3 \\ 10 & 3 & 7 & 6 \\ 0 & 20 & 13 & 5 \end{pmatrix}$$

Row reduction:

$$\begin{pmatrix} 5 & -1 & 2 & 3 \\ 10 & 3 & 7 & 6 \\ 0 & 20 & 13 & 5 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 5 & -1 & 2 & 3 \\ 0 & 5 & 3 & 0 \\ 0 & 20 & 13 & 5 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 5 & -1 & 2 & 3 \\ 0 & 5 & 3 & 0 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

U

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 4 & 1 \end{pmatrix}$$

6. The matrix

$$A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$$

has the LU factorization $A = LU$ where

$$L = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \text{ and } U = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Use this factorization to solve

$$Ax = \begin{pmatrix} 1 \\ -4 \end{pmatrix}.$$

Show clearly the two steps.

① $Ly = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$

$$\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 &= -4 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} y_1 &= 1 \\ y_2 &= -6 \end{aligned}$$

② $Ux = \begin{pmatrix} 1 \\ -6 \end{pmatrix}$

$$\begin{aligned} x_1 + x_2 &= 1 \\ -x_2 &= -6 \end{aligned} \quad \rightsquigarrow \quad \begin{aligned} x_1 &= -5 \\ x_2 &= 6 \end{aligned}$$

$$\begin{pmatrix} -5 \\ 6 \end{pmatrix}$$

7. Let A be an $n \times n$ matrix. Which of the following statements are equivalent to the statement "A is invertible"? Circle all that apply.

(i) the columns of A span $\mathbb{R}^n$

(ii) the rows of A span $\mathbb{R}^n$

(iii) the equation $Ax = 0$ has the trivial solution

(iv) the equation $Ax = 0$ has infinitely many solutions

(v) the equation $Ax = b$ is consistent for all b in $\mathbb{R}^n$

(vi) the equation $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$

(vii) the rank of A is n

(viii) the dimension of the null space of A is 0

(ix) A is equal to a product of elementary matrices

(x) A^5 is invertible

8. Suppose that A is a 2×2 matrix and that

$$Ax = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$$

has exactly one solution. Is the equation

$$Ax = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$$

consistent? Answer *yes/no/maybe* and explain.

Yes. If $Ax = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ has one soln then

$Ax = 0$ has one soln

so A is invertible so every $Ax = b$

is consistent.

Suppose that A is a 2×2 matrix with two identical columns. Is the equation

$$Ax = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$$

consistent? Answer *yes/no/maybe* and explain.

Maybe:

Yes if $A = \begin{pmatrix} 7 & 7 \\ 3 & 3 \end{pmatrix}$

No if $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

9. Determine if the following matrix is invertible and, if so, find the inverse.

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 2 & 0 & 11 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 11 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & 0 & -5 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 11 & 0 & -5 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$$

10. What is the inverse of the matrix

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}?$$

$$\begin{pmatrix} 1 & -2 \\ -2 & 5 \end{pmatrix}$$

Suppose that A and B are square matrices and that B is the inverse of A^2. Express the inverse of A in terms of A and B.

$$B^{-1} = A^2$$

$$A^{-1}B^{-1} = A$$

$$A^{-1} = AB$$