Name Prof M

Section HP ___

Mathematics 1553

Midterm 3

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1. Define *eigenvector.*

An eigenvector for a matrix A is a nonzero vector v so that Av is equal to a multiple of v.

Define *diagonalizable.*

A matrix is diagonalizable if it is similar to a diagonal matrix

2. Suppose that A is a 2×2 matrix and the associated linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is orthogonal projection onto the y-axis. List the eigenvalues of A (if there are any) and give a basis for each corresponding eigenspace.

eigenval.	basis for eigensp.
1	$\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$
0	$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

Suppose that A is a 2×2 matrix and that the associated linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation about the origin by $\pi/4$. List the eigenvalues of A (if there are any) and give a basis for each corresponding eigenspace.

No (real) eigenvalues

3. Answer *yes/no/maybe* for each question.

Suppose A is a 2×2 matrix that is row equivalent to the identity. Is A diagonalizable?

Maybe

Yes Example: $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

No Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Suppose A is a 2×2 matrix with two distinct eigenvalues. Is A invertible?

Maybe

Yes Example: $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$

No Example: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

Suppose A is a 2×2 matrix with only one eigenvalue, which is 1. Is A diagonalizable?

Maybe

Yes Example: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

No Example: $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

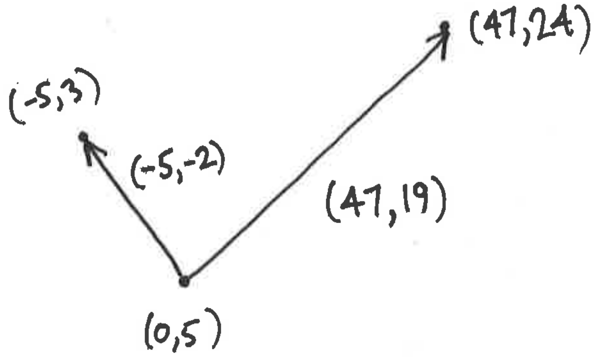
Suppose A is an $n \times n$ matrix that is similar to the identity matrix. Is A diagonalizable?

Yes

Suppose A is a 5×5 matrix with eigenvalues 1, 2, 3, and 4 and the dimension of the eigenspace for the eigenvalue 3 is 2. Is A diagonalizable?

Yes

4. What is the area of the parallelogram with vertices $(-5, 3)$, $(0, 5)$, $(47, 24)$, and $(52, 26)$?



$$\det\begin{pmatrix} -5 & 47 \\ -2 & 19 \end{pmatrix} = -95 + 94 = -1$$

$$\boxed{1}$$

Compute the determinant of the following matrix:

$$\begin{pmatrix} 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

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5. Suppose that A is a 5×5 matrix with determinant 3.

Is A invertible?

yes

What is $\det(-A)$?

$(-1)^5 \cdot 3 = -3$

What is $\det(A^{-1})$?

$1/3$

What is $\det A^T$?

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What is the determinant of the matrix obtained from A by replacing the first row with twice the first row plus the second row?

$2 \cdot 3 = 6$

6. Consider the following matrix:

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$$

Is A diagonalizable? If so, diagonalize it. If not, explain why not.

eigenvalues : $\det \begin{pmatrix} 1-\lambda & -1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2 - \lambda = 0$

$\lambda = 0, \lambda = 1$

0-eigenvector : $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x_1 - x_2 = 0 \\ 0 = 0 \end{matrix} \rightsquigarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

1-eigenvector : $\begin{pmatrix} 0 & -1 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow x_2 = 0 \rightsquigarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$$

7. Consider the following matrix:

$$A = \begin{pmatrix} 7 & -6 \\ 1 & 2 \end{pmatrix}$$

which satisfies

$$A = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

Give a formula for A^{100}. Your answer should be a single matrix, but the entries do not need to be simplified.

$$A^{100} = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^{100} & 0 \\ 0 & 4^{100} \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 5^{100} & -2\cdot 5^{100} \\ -4^{100} & 3\cdot 4^{100} \end{pmatrix}$$

$$= \begin{pmatrix} 3\cdot 5^{100} - 2\cdot 4^{100} & -6\cdot 5^{100} + 6\cdot 4^{100} \\ 5^{100} - 4^{100} & -2\cdot 5^{100} + 3\cdot 4^{100} \end{pmatrix}$$

Which describes the linear transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$?

(a) It stretches $(3, 1)$ by 4 and $(2, 1)$ by 5.

(b) It stretches $(1, -1)$ by 4 and $(-2, 3)$ by 5.

(c) It stretches $(3, 1)$ by 5 and $(2, 1)$ by 4.

(d) It stretches $(1, -1)$ by 5 and $(-2, 3)$ by 4.

What is the limit of the slope of $A^k(e_1)$ as k tends to infinity?

(a) 1/3 $\leftarrow$ Slope of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(b) 2

(c) 1/2

(d) $-1/2$

8. Find the eigenvalues of the following matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

Hint: When finding the characteristic polynomial, resist the urge to multiply everything out.

$$\det(A - \lambda I) = 0$$

$$\det\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & 1 & -1-\lambda \end{pmatrix} = (1-\lambda)\left[(2-\lambda)(-1-\lambda)+1\right] + \left[(-1)(-1-\lambda)+0\right] = 0$$

$$= (1-\lambda)\left[-2-2\lambda+\lambda+\lambda^2+1\right] + \left[1+\lambda\right] = 0$$

$$= (1-\lambda)\left[\lambda^2-\lambda-1\right] + \left[1+\lambda\right] = 0$$

$$= \lambda^2 \cancel{-\lambda} \cancel{-1} - \lambda^3 + \lambda^2 + \lambda \cancel{+1} \cancel{+\lambda} = 0$$

$$-\lambda^3 + 2\lambda^2 + \lambda = 0$$

$$-\lambda(\lambda^2 - 2\lambda - 1) = 0$$

$\lambda = 0$

$$\lambda = \frac{2 \pm \sqrt{4+4}}{2} = \frac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$$

$$\lambda = 0,\ 1 \pm \sqrt{2}$$

9. Find the adjugate of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Remember that the adjugate of a matrix is the matrix whose ijth entry is the jith cofactor of A.

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

What is A^{-1}?

$$\begin{pmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

10. Let B be a ball of radius 1 in $\mathbb{R}^3$. The volume of B is $4\pi/3$. Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 9 \\ 0 & 0 & 2 \end{pmatrix}$$

What is the volume of $T_A(B)$?

$\det A \cdot \mathrm{vol}(B)$

$= 24 \cdot 4\pi/3 = 32\pi$

Suppose that A is a matrix where the entries in each column add up to 1 (for example, the matrices in the Google Pagerank algorithm). Show that A has an eigenvalue equal to 1. *Hint: find an eigenvector for A^T.*

$$A^T \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\Rightarrow$ A^T has eigenvalue 1

$\Rightarrow$ A has eigenvalue 1.