MATH 1553
SAMPLE MIDTERM 1: THROUGH 1.5

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- Each problem is worth 10 points. The maximum score on this exam is 50 points.
- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

Problem 1. [2 points each]

In this problem, A is an $m \times n$ matrix (m rows and n columns) and b is a vector in $\mathbf{R}^m$. Circle **T** if the statement is always true (for any choices of A and b) and circle **F** otherwise. Do not assume anything else about A or b except what is stated.

a) **T** **F** The matrix below is in reduced row echelon form.

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & -3 & 1 \\ 0 & 0 & 1 & -1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

b) **T** **F** If A has fewer than n pivots, then $Ax = b$ has infinitely many solutions.

c) **T** **F** If the columns of A span $\mathbf{R}^m$, then $Ax = b$ must be consistent.

d) **T** **F** If $Ax = b$ is consistent, then the equation $Ax = 5b$ is consistent.

e) **T** **F** If $Ax = b$ is consistent, then the solution set is a span.

Solution.

a) True.

b) False: For example, $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has one pivot but has no solutions.

c) True: the span of the columns of A is exactly the set of all v for which $Ax = v$ is consistent. Since the span is $\mathbf{R}^m$, the matrix equation is consistent no matter what b is.

d) True: If $Aw = b$ then $A(5w) = 5Aw = 5b$.

e) False: it is a *translate* of a span (unless $b = 0$).

Problem 2. [5 points each]

Acme Widgets, Gizmos, and Doodads has two factories. Factory A makes 10 widgets, 3 gizmos, and 2 doodads every hour, and factory B makes 4 widgets, 1 gizmo, and 1 doodad every hour.

a) If factory A runs for a hours and factory B runs for b hours, how many widgets, gizmos, and doodads are produced? Express your answer as a vector equation.

b) A customer places an order for 16 widgets, 5 gizmos, and 3 doodads. Can Acme fill the order with no widgets, gizmos, or doodads left over? If so, how many hours do the factories run? If not, why not?

Solution.

a) Let w, g, and d be the number of widgets, gizmos, and doodads produced.

$$\begin{pmatrix} w \\ g \\ d \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

b) We need to solve the vector equation

$$\begin{pmatrix} 16 \\ 5 \\ 3 \end{pmatrix} = a \begin{pmatrix} 10 \\ 3 \\ 2 \end{pmatrix} + b \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}.$$

We put it into an augmented matrix and row reduce:

$$\left(\begin{array}{cc|c} 10 & 4 & 16 \\ 3 & 1 & 5 \\ 2 & 1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 3 & 1 & 5 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 2 & 1 & 3 \\ 10 & 4 & 16 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 10 & 4 & 16 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

These equations are consistent, but they tell us that factory B would have to run for -1 hours! Therefore it can't be done.

Problem 3. [10 points]

Consider the system below, where h and k are real numbers.

$$x + 3y = 2$$
$$3x - hy = k.$$

a) Find the values of h and k which make the system inconsistent.

b) Find the values of h and k which give the system a unique solution.

c) Find the values of h and k which give the system infinitely many solutions.

Solution.

We form an augmented matrix and row-reduce.

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ 3 & -h & k \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & -h-9 & k-6 \end{array}\right)$$

a) The system is inconsistent precisely when the augmented matrix has a pivot in the rightmost column. This is when $-h - 9 = 0$ and $k - 6 \neq 0$, so $\boxed{h = -9 \text{ and } k \neq 6}$.

b) The system has a unique solution if and only if the left two columns are pivot columns. We know the first column has a pivot, and the second column has a pivot precisely when $-h - 9 \neq 0$, so $\boxed{h \neq -9 \text{ and } k \text{ can be any real number}}$.

c) The system has infinitely many solutions when the system is consistent and has a free variable (which in this case must be y), so $-h - 9 = 0$ and $k - 6 = 0$, hence $\boxed{h = -9 \text{ and } k = 6}$.

Problem 4.

[10 points]

Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) [4 points] Find the parametric vector form for the general solution.

b) [3 points] Find the parametric vector form of the corresponding *homogeneous* equations.

c) [3 points] Unrelated to parts (a) and (b).
If b, v, w are vectors in $\mathbf{R}^3$ and $\text{Span}\{b, v, w\} = \mathbf{R}^3$, is it possible that b is in $\text{Span}\{v, w\}$? Fully justify your answer.

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 3 & 4 & 5 & 6 & -2 \\ 5 & 6 & 7 & 8 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & -2 & -4 & -6 & 4 \\ 0 & -4 & -8 & -12 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_3 and x_4 are free, and the general solution is

$$\begin{aligned} x_1 \qquad - x_3 - 2x_4 &= 2 \\ x_2 + 2x_3 + 3x_4 &= -2 \end{aligned} \implies \begin{aligned} x_1 &= x_3 + 2x_4 + 2 \\ x_2 &= -2x_3 - 3x_4 - 2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

This gives the parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\text{Span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\} \quad \text{by} \quad \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\text{Span}\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix}, \begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}.$$

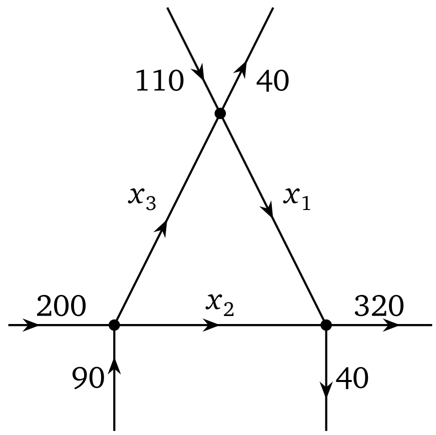
c) No. Recall that Span$\{b, v, w\}$ is the set of all linear combinations of b, v, and w. If b is in Span$\{v, w\}$ then b is a linear combination of v and w. Consequently, any element of Span$\{b, v, w\}$ is a linear combination of v and w and is therefore in Span$\{v, w\}$, which is at most a 2-plane and cannot be all of $\mathbf{R}^3$.

To see why the span of v and w can never be $\mathbf{R}^3$, consider the matrix A whose columns are v and w. Since A is 3×2, it has at most two pivots, so A cannot have a pivot in every row. Therefore, by a theorem from section 1.4, the equation $Ax = b$ will fail to be consistent for some b in $\mathbf{R}^3$, which means that some b in $\mathbf{R}^3$ is not in the span of v and w.

Problem 5. [10 points]

The diagram below describes traffic in a part of town.

Traffic flow (cars/hr)



a) Write a system of three linear equations in x_1, x_2, and x_3 corresponding to the traffic flow.

b) Use an augmented matrix to solve this system of linear equations. Were we given enough information to know the exact values of x_1, x_2, and x_3?

Solution.

a) For the top, bottom right, and bottom left nodes, the number of cars entering must match the number of cars exiting, so the system is:

$$x_1 + 40 = x_3 + 110$$
$$x_1 + x_2 = 360$$
$$x_2 + x_3 = 290.$$

b) The system can be written

$$x_1 - x_3 = 70$$
$$x_1 + x_2 = 360$$
$$x_2 + x_3 = 290.$$

We form an augmented matrix and perform row operations.

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 70 \\ 1 & 1 & 0 & 360 \\ 0 & 1 & 1 & 290 \end{array}\right) \xrightarrow{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 70 \\ 0 & 1 & 1 & 290 \\ 0 & 1 & 1 & 290 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 70 \\ 0 & 1 & 1 & 290 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, x_3 is a free variable, $x_1 = x_3 + 70$, and $x_2 = 290 - x_3$.

We cannot know the exact values of x_1, x_2, and x_3 with the information we have only been given. For example, we could have $x_3 = 0$, $x_2 = 290$, $x_1 = 70$. Or, we could have $x_3 = 100$, $x_2 = 190$, $x_1 = 170$, etc.

[Scratch work]