MATH 1553
SAMPLE MIDTERM 2: 1.7-1.9, 2.1-2.3, 2.8-2.9

Name		**Section**	

1	2	3	4	5	Total

Please **read all instructions** carefully before beginning.

- You have 50 minutes to complete this exam.
- There are no aids of any kind (notes, text, etc.) allowed.
- Please show your work.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is intended to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. I recommend completing the practice exam in 50 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §1.7 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§1.7 through 2.9.

Problem 1.

[2 points each]

Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) **T** **F** If A is an $n \times n$ matrix and its rows are linearly independent, then $Ax = b$ has a unique solution for every b in $\mathbf{R}^n$.

b) **T** **F** If A is an $n \times n$ matrix and $Ae_1 = Ae_2$, then A is not invertible.

c) **T** **F** The solution set of a consistent matrix equation $Ax = b$ is a subspace.

d) **T** **F** If A and B are matrices and AB is invertible, then A and B are invertible.

e) **T** **F** There exists a 3×5 matrix with rank 4.

Solution.

a) True: A has n pivots, so A is invertible (by Inv. Mat. Thm.), thus $Ax = b$ is consistent and has a unique solution for every b in $\mathbf{R}^n$.

b) True: $x \to Ax$ is not one-to-one, so A is not invertible.

c) False: this is true if and only if $b = 0$, i.e., the equation is *homogeneous*, in which case the solution set is the null space of A.

d) False: Take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an invertible 2×2 matrix, but A and B are not invertible (they are not even square matrices!).

e) False: the rank is the dimension of the column space, which is a subspace of $\mathbf{R}^3$, hence has dimension at most 3.

Problem 2.

Parts (a) and (b) are unrelated.

a) Find all values of x so that the matrix $\begin{pmatrix} -1 & 2-x \\ x & 3 \end{pmatrix}$ is invertible.

b) Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+z \\ 3x-4y+z \end{pmatrix}$. Is T invertible? Justify your answer.

Solution.

a) A matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible unless $ad - bc = 0$. Therefore, $\begin{pmatrix} -1 & 2-x \\ x & 3 \end{pmatrix}$ is invertible unless

$$-3 - x(2-x) = 0, \qquad x^2 - 2x - 3 = 0, \qquad (x-3)(x+1) = 0.$$

Therefore, the matrix is invertible as long as $x \neq 3$ and $x \neq -1$.

b) One approach: We form the standard matrix A for T:

$$A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \end{pmatrix}.$$

We row-reduce A until we determine its pivot columns

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 3 & -4 & 1 \end{pmatrix} \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 - R_1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

A has a pivot in every column, so A and therefore T are invertible by the Invertible Matrix Theorem.

Alternative approach: T is a linear transformation from $\mathbf{R}^3$ to $\mathbf{R}^3$, so by the Invertible Matrix Theorem, it is invertible if and only if it is one-to-one, if and only if $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has only the trivial solution.

If $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ x+z \\ 3x-4y+z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ then $x = 0$, and

$$x + z = 0 \implies 0 + z = 0 \implies z = 0, \text{ and finally}$$

$$3x - 4y + z = 0 \implies 0 - 4y + 0 = 0 \implies y = 0,$$

so the trivial solution $x = y = z = 0$ is the only solution the homogeneous equation. Therefore, T is invertible.

Problem 3.

a) Determine which of the following transformations are linear.

(1) $S : \mathbf{R}^2 \to \mathbf{R}^2$ given by $S(x_1, x_2) = (x_1,\ 3 + x_2)$

(2) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1 - x_2,\ x_1 x_2)$

(3) $U : \mathbf{R}^2 \to \mathbf{R}^3$ given by $U(x_1, x_2) = (-x_2,\ x_1, 0)$

b) Complete the following definition (be mathematically precise!):
A set of vectors $\{v_1, v_2, \dots, v_p\}$ in $\mathbf{R}^n$ is *linearly independent* if...

c) If $\{v_1, v_2, v_3\}$ are vectors in $\mathbf{R}^3$ with the property that none of the vectors is a scalar multiple of another, is $\{v_1, v_2, v_3\}$ necessarily linearly independent? Justify your answer.

Solution.

a) (1) S is not linear: $S\big((1,0) + (1,0)\big) = (2,3)$ but $S(1,0) + S(1,0) = (2,6)$.

(2) T is not linear: $T(1,1) + T(1,1) = (0,2)$, but $2T(1,1) = (0,4)$.

(3) U is linear.

b) the vector equation $x_1 v_1 + x_2 v_2 + \cdots + x_p v_p = 0$ has only the trivial solution $x_1 = x_2 = \cdots = x_p = 0$.

c) No. For example, take $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

No vector in the set is a scalar multiple of any other, but nonetheless $\{v_1, v_2, v_3\}$ is linearly dependent. In fact, $v_3 = v_1 + v_2$.

Problem 4.

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation which projects onto the yz-plane and then forgets the x-coordinate, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation counterclockwise by $60°$. Their standard matrices are

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

respectively.

a) Which inverse makes sense / exists? (Circle one.)

$$T^{-1} \qquad U^{-1}$$

b) Find the standard matrix for the inverse you circled in (a).

c) Which composition makes sense? (Circle one.)

$$U \circ T \qquad T \circ U$$

d) Find the standard matrix for the transformation that you circled in (c).

Solution.

a) Only U has the same domain and codomain, so U^{-1} makes sense.

b) The standard matrix for U^{-1} is B^{-1}. We have $\det B = 1$, so

$$B^{-1} = \frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix}.$$

c) Only $U \circ T$ makes sense, as the codomain of T is $\mathbf{R}^2$, which is the domain of U.

d) The standard matrix for $U \circ T$ is

$$BA = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 1 & -\sqrt{3} \\ 0 & \sqrt{3} & 1 \end{pmatrix}.$$

Problem 5.

Consider the following matrix A and its reduced row echelon form:

$$\begin{pmatrix} 2 & 4 & 7 & -16 \\ 3 & 6 & -1 & -1 \\ 5 & 10 & 6 & -17 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Find a basis for Col$A$.

b) Find a basis $\mathcal{B}$ for Nul$A$.

c) For each of the following vectors v, decide if v is in Nul$A$, and if so, find $[x]_{\mathcal{B}}$:

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \qquad \begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix}$$

Solution.

a) The pivot columns for A form a basis for Col$A$, so a basis is $\left\{ \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ -1 \\ 6 \end{pmatrix} \right\}$.

b) We compute the parametric vector form for the general solution of $Ax = 0$:

$$\begin{aligned} x_1 &= -2x_2 + x_4 \\ x_2 &= x_2 \\ x_3 &= 2x_4 \\ x_4 &= x_4 \end{aligned} \rightsquigarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix}.$$

Therefore, a basis is given by

$$\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$$

c) First we note that if

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix},$$

then $c_1 = b$ and $c_2 = d$. This makes it easy to check whether a vector is in Nul$A$, and to compute the $\mathcal{B}$-coordinates.

$$\begin{pmatrix} 7 \\ 3 \\ 1 \\ 2 \end{pmatrix} \neq 3 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \text{not in Nul}A.$$

$$\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix} = 2\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 2 \\ 1 \end{pmatrix} \implies \left[\begin{pmatrix} -5 \\ 2 \\ -2 \\ -1 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

[Scratch work]