Math 1553 Worksheet: Lines and planes in $\mathbb{R}^n$ and §1.1

Solutions

1. Which of the following equations are linear? Justify your answers.

a) $3x_1 + \sqrt{x_2} = 4$

b) $x_1 = x_2 - x_3 + 10x_4$.

c) $\pi x + \ln(13)y + z = \sqrt[3]{2}$

Solution.

a) No. The $\sqrt{x_2}$ term makes it non-linear.

b) Yes.

c) Yes. The $\sqrt[3]{2}$ term is just a constant. Don't be misled by the appearance of the natural logarithm: $\ln(13)$ is just the coefficient for y.

If the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Find all values of h so that the lines $x + hy = -5$ and $2x - 8y = 6$ do *not* intersect.

Solution.

We can use standard algebra, row operations, or geometric intuition.

Using standard algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$x + hy = -5$$
$$2x - 8y = 6.$$

Subtracting twice the first equation from the second equation gives us

$$x + hy = -5$$
$$0 + (-8 - 2h)y = 16.$$

If $-8 - 2h = 0$ (so $h = -4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h = -4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8 - 2h)y = 16 \qquad y = \frac{16}{-8 - 2h} \qquad y = \frac{8}{-4 - h}.$$

We can now substitute this value of y into the first equation to find x:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4 - h} = -5 \qquad x = -5 - \frac{8h}{-4 - h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h = -4}$.

Using row operations: Like the previous technique, let's see what happens if the lines intersect. We put the equations into augmented matrix form and use row operations.

$$\left[\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right] \xrightarrow{R_2=R_2-2R_1} \left[\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right].$$

If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y=16$, so $y=\dfrac{16}{-8-2h}=\dfrac{8}{-4-h}$, and $x=-5-hy=-5-\dfrac{8h}{-4-h}$, so the lines intersect at $\left(-5-\dfrac{8h}{-4-h},\ \dfrac{8}{-4-h}\right)$.

Therefore, our answer is $h=-4$.

Using intuition from geometry: Two non-identical lines in the xy-plane intersect if and only if they are not parallel. The second line is $y=\dfrac{1}{4}x-\dfrac{3}{4}$, so its slope is $\dfrac{1}{4}$. If $h \neq 0$, then the first line is $y=-\dfrac{1}{h}x-\dfrac{5}{h}$, so the lines are parallel when $-\dfrac{1}{h}=\dfrac{1}{4}$, which means $h=-4$. You can check that when $h=-4$ the lines aren't identical. (And if $h=0$ then the first line is vertical so it isn't parallel to the second).

3. For each of the following, answer true or false. Justify your answer.

a) Every system of linear equations has at least one solution.

b) There is a system of linear equations that has exactly 5 solutions.

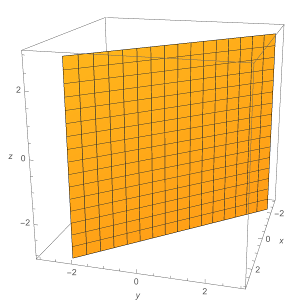
c) If a, b, and c are real numbers, then the equation $ax+by=c$ for (x,y,z) in $\mathbf{R}^3$ describes a line.

Solution.

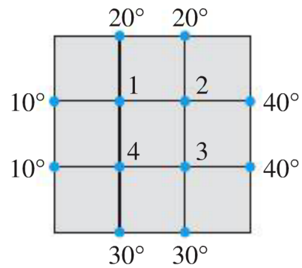
a) False. Some examples from class and this worksheet have no solutions.

b) False. There are only three possibilities: no solutions, exactly one solution, or infinitely many solutions.

c) False. For example, in $\mathbf{R}^3$, the equation $x+y=1$ corresponds geometrically to a vertical plane. We could write the plane in parametric form as $(t,\ 1-t,\ z)$ where t and z vary among all real numbers.



4. The picture below represents the temperatures at four interior nodes of a mesh.



Let $T_1, \ldots, T_4$ be the temperatures at nodes 1 through 4. Suppose that the temperature at each node is the average of the four nearest nodes. For example,

$$T_1 = \frac{10 + 20 + T_2 + T_4}{4}.$$

a) Write a system of four linear equations whose solution would give the temperatures $T_1, \ldots, T_4$.

b) Write an augmented matrix that represents that system of equations.

Solution.

(a)

The first equation was given. The others are:

$T_2 = (T_1 + 20 + 40 + T_3)/4$, or $4T_2 - T_1 - T_3 = 60$

$T_3 = (T_4 + T_2 + 40 + 30)/4$, or $4T_3 - T_4 - T_2 = 70$

$T_4 = (10 + T_1 + T_3 + 30)/4$, or $4T_4 - T_1 - T_3 = 40$

(b) To put this in matrix form, we need to put everything in order.

$$\begin{array}{rcrcrcrcl} 4T_1 & - & T_2 & & & - & T_4 & = & 30 \\ -T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\ & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\ -T_1 & & & - & T_3 & + & 4T_4 & = & 40 \end{array}$$

This gives the augmented matrix

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix}.$$

5. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ 2x + 5y + 2z &= -1 \\ y + 3z &= 8. \end{aligned}$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

We can isolate z in the third equation using algebra, but it is probably best to do using an augmented matrix and elementary row operations.

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right] \xrightarrow{R_2 = R_2 - R_1} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right] \xrightarrow{R_3 = R_3 - R_2} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right].$$

The last line is $5z = 10$, so $z = 2$. Since we don't have much practice with row-reduction, we will use substitution to finish.

The second equation is $y - 2z = -2$, so $y - 2(2) = -2$, thus $y = 2$.

The first equation is $2x + 4(2) + 4(2) = 1$, so $2x = -15$, thus $x = -\frac{15}{2}$.

We have found that the planes intersect at the point $\left(-\frac{15}{2},\ 2,\ 2\right)$.