Math 1553 Worksheet §5.3, 5.5

1. Answer yes / no / maybe. In each case, $A$ is a matrix whose entries are real.
   
   a) If $A$ is a $3 \times 3$ matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the 5-eigenspace is 2-dimensional.
   
   b) If $A$ is an invertible $2 \times 2$ matrix, then $A$ is diagonalizable.
   
   c) Can a $3 \times 3$ matrix $A$ have a non-real complex eigenvalue with multiplicity 2?
   
   d) Can a $3 \times 3$ matrix $A$ have eigenvalues 3, 5, and $2 + i$?

Solution.

a) Maybe. The geometric multiplicity of $\lambda = 5$ can be 1 or 2. For example, the matrix
   
   \[
   \begin{pmatrix}
   5 & 0 & 0 \\
   0 & 5 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \]
   
   has a 5-eigenspace which is 2-dimensional, whereas the matrix
   
   \[
   \begin{pmatrix}
   5 & 1 & 0 \\
   0 & 5 & 0 \\
   0 & 0 & 0
   \end{pmatrix}
   \]
   
   has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(\lambda - 5)^2$.
   
   b) Maybe. The identity matrix is invertible and diagonalizable, but the matrix
   
   \[
   \begin{pmatrix}
   1 & 1 \\
   0 & 1
   \end{pmatrix}
   \]
   
   is invertible but not diagonalizable.
   
   c) No. If $c$ is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\overline{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean $A$ has a characteristic polynomial of degree 4 or more, which is impossible for a $3 \times 3$ matrix.
   
   d) No. If $2 + i$ is an eigenvalue then so is its conjugate $2 - i$.

2. Let $A = \begin{pmatrix} 8 & 36 & 62 \\ -6 & -34 & -62 \\ 3 & 18 & 33 \end{pmatrix}$.
   
   The characteristic polynomial for $A$ is $-\lambda^3 + 7\lambda^2 - 16\lambda + 12$, and $\lambda - 3$ is a factor. Decide if $A$ is diagonalizable. If it is, find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$.

Solution.

By polynomial division,

\[
\frac{-\lambda^3 + 7\lambda^2 - 16\lambda + 12}{\lambda - 3} = -\lambda^2 + 4\lambda - 4 = -(\lambda - 2)^2.
\]
Thus, the characteristic poly factors as \(-(\lambda-3)(\lambda-2)^2\), so the eigenvalues are \(\lambda_1 = 3\) and \(\lambda_2 = 2\).

For \(\lambda_1 = 3\), we row-reduce \(A - 3I\):

\[
\begin{pmatrix}
  5 & 36 & 62 \\
-6 & -37 & -62 \\
 3 & 18 & 30
\end{pmatrix}
\xrightarrow{R_1 \leftrightarrow R_3}
\begin{pmatrix}
  1 & 6 & 10 \\
-6 & -37 & -62 \\
 5 & 36 & 62
\end{pmatrix}
\xrightarrow{\text{New } R_1 / 3}
\begin{pmatrix}
  1 & 6 & 10 \\
0 & -1 & -2 \\
0 & 6 & 12
\end{pmatrix}
\]

\[
\xrightarrow{R_3 = R_3 + 6R_2}
\begin{pmatrix}
  1 & 6 & 10 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix}
\xrightarrow{R_1 = R_1 - 6R_2}
\begin{pmatrix}
  1 & 0 & -2 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{pmatrix}.
\]

Therefore, the solutions to \((A - 3I \mid 0)\) are \(x_1 = 2x_3, \ x_2 = -2x_3, \ x_3 = x_3\).

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= \begin{pmatrix}
  2x_3 \\
-2x_3 \\
 x_3
\end{pmatrix}
= x_3 \begin{pmatrix}
  2 \\
-2 \\
 1
\end{pmatrix}.
\]

The 3-eigenspace has basis \(\left\{ \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} \right\}\).

For \(\lambda_2 = 2\), we row-reduce \(A - 2I\):

\[
\begin{pmatrix}
  6 & 36 & 62 \\
-6 & -36 & -62 \\
 3 & 18 & 31
\end{pmatrix}
\xrightarrow{\text{rref}}
\begin{pmatrix}
  1 & 0 & \frac{31}{3} \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}.
\]

The solutions to \((A - 2I \mid 0)\) are \(x_1 = -6x_2 - \frac{31}{3}x_3, \ x_2 = x_2, \ x_3 = x_3\).

\[
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
= \begin{pmatrix}
-6x_2 - \frac{31}{3}x_3 \\
 x_2 \\
 x_3
\end{pmatrix}
= x_2 \begin{pmatrix}
  -6 \\
 1 \\
 0
\end{pmatrix}
+ x_3 \begin{pmatrix}
  \frac{-31}{3} \\
 0 \\
 1
\end{pmatrix}.
\]

The 2-eigenspace has basis \(\left\{ \begin{pmatrix} -6 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{-31}{3} \\ 0 \\ 1 \end{pmatrix} \right\}\).

Therefore, \(A = PDP^{-1}\) where

\[
P = \begin{pmatrix}
  2 & -6 & \frac{-31}{3} \\
-2 & 1 & 0 \\
 1 & 0 & 1
\end{pmatrix}
\quad D = \begin{pmatrix}
  3 & 0 & 0 \\
 0 & 2 & 0 \\
 0 & 0 & 2
\end{pmatrix}.
\]

Note that we arranged the eigenvectors in \(P\) in order of the eigenvalues 3, 2, 2, so we had to put the diagonals of \(D\) in the same order.
3. Let \( A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \).

   a) Find all eigenvalues and eigenvectors of \( A \).

   b) Write \( A = PCP^{-1} \), where \( C \) is a rotation followed by a scale. Describe what \( A \) does geometrically. Draw a picture.

**Solution.**

a) The characteristic polynomial is
   \[
   \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5
   \]
   \[
   \lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4 - 20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.
   \]
   For the eigenvalue \( \lambda = 1 - 2i \), we row-reduce \((A - (1 - 2i)I)\begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 - i \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).
   So \( x_1 = ix_2 \) and \( x_2 = x_2 \). A corresponding eigenvector is \( v = \begin{pmatrix} i \\ 1 \end{pmatrix} \), and any nonzero complex multiple of \( v \) will also be an eigenvector.
   (If we used the \( 2 \times 2 \) trick from the 5.5 slides, we would have found that an eigenvector is \( \begin{pmatrix} 2 \\ -2i \end{pmatrix} \), which is really just \( -2i \) times the eigenvector \( v \) above.)

   From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue \( \lambda = 1 + 2i \), a corresponding eigenvector is \( w = \bar{v} = \begin{pmatrix} -i \\ 1 \end{pmatrix} \).

b) We use \( \lambda = 1 - 2i \) and its associated \( v = \begin{pmatrix} i \\ 1 \end{pmatrix} \).

   \[ A = PCP^{-1} \text{ where } P = \begin{pmatrix} \text{Re}(v) & \text{Im}(v) \\ 0 & 1 \end{pmatrix} \text{ and } \]
   \[ C = \begin{pmatrix} \text{Re}(\lambda) & \text{Im}(\lambda) \\ -\text{Im}(\lambda) & \text{Re}(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}. \]

   The scale is by a factor of \( |\lambda| = |1 + 2i| = \sqrt{1^2 + 2^2} = \sqrt{5} \). If we factor this out of \( C \) we get
   \[ C = \sqrt{5} \begin{pmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}. \]

   We see \( \cos(\theta) = \frac{1}{\sqrt{5}} \) and \( \sin(\theta) = \frac{2}{\sqrt{5}} \), so \( \tan(\theta) = 2 \) and \( \theta = \arctan(2) \).

   \( C \) is rotation by the angle \( \arctan(2) \), followed by scaling by a factor of \( \sqrt{5} \).

   See the [interactive] demo for how \( A \) acts geometrically.
For example, the $2 \times 2$ trick from the 5.5 slides says that if $\lambda$ is an eigenvalue of $A$, then one eigenvector is $\begin{pmatrix} b \\ -a \end{pmatrix}$ where $\begin{pmatrix} a & b \end{pmatrix}$ is the first row of $A - \lambda I$.

Row 1 of $A - \lambda I$ was $\begin{pmatrix} 2i & 2 \end{pmatrix}$, so $\begin{pmatrix} 2 \\ -2i \end{pmatrix}$ as an eigenvector.

This would give us $P = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$ rather than $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. However, it would still be the case that $A = PCP^{-1}$ since

$$PCP^{-1} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} = A.$$ 

### Supplemental Problems

For those who want additional practice problems after completing the worksheet, here are some extra practice problems.

1. Let $A$ and $B$ be $3 \times 3$ real matrices. Answer yes / no / maybe:
   a) If $A$ and $B$ have the same eigenvalues, then $A$ is similar to $B$.
   b) If $A$ and $B$ both have eigenvalues $-1, 0, 1$, then $A$ is similar to $B$.
   c) If $A$ is diagonalizable and invertible, then $A^{-1}$ is diagonalizable.

### Solution.

a) Maybe. For example, $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ have the same eigenvalues ($\lambda = 0$ with alg. multiplicity 2) but are not similar, whereas $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is similar to itself.

b) Yes. In this case, $A$ and $B$ are $3 \times 3$ matrices with 3 distinct eigenvalues and thus automatically diagonalizable, and each is similar to $D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Since $A$ and $D$ are similar, and $B$ and $D$ are similar, it follows that $A$ and $B$ are similar.

$$A = PDP^{-1} \quad B = QDQ^{-1} \quad A = PDP^{-1} = PQ^{-1}BQ^{-1} = PQ^{-1}B(PQ^{-1})^{-1}.$$ 

c) Yes. If $A = PDP^{-1}$ and $A$ is invertible then its eigenvalues are all nonzero, so the diagonal entries of $D$ are nonzero and thus $D$ is invertible (pivot in every diagonal position). Thus, $A^{-1} = (PDP^{-1})^{-1} = (P^{-1})^{-1}D^{-1}P^{-1} = PD^{-1}P^{-1}$. 

2. Give an example of a non-diagonal $2 \times 2$ matrix which is diagonalizable but not invertible. Justify your answer.

Solution.

\[
\begin{pmatrix}
1 & 1 \\
0 & 0
\end{pmatrix}
\]
is not invertible (row of zeros) but is diagonalizable since its has two distinct eigenvalues 0 and 1 (it is triangular, so its diagonals are its eigenvalues).

3. Suppose $A$ is a $7 \times 7$ matrix with four distinct eigenvalues. One eigenspace has dimension 2, while another eigenspace has dimension 3. Is it possible that $A$ is not diagonalizable?

Solution.

$A$ must be diagonalizable. It is a general fact that every eigenvalue of a matrix has a corresponding eigenspace which is at least 1-dimensional. Given this and the fact that $A$ has four total eigenvalues, we see the sum of dimensions of the eigenspaces of $A$ is at least $2 + 3 + 1 + 1 = 7$, and in fact must equal 7 since that is the max possible for a $7 \times 7$ matrix. Therefore, $A$ has 7 linearly independent eigenvectors and is therefore diagonalizable.

4. Let $A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$.

a) Find all (complex) eigenvalues and eigenvectors of $A$.

b) Write $A = PCP^{-1}$, where $C$ is a block diagonal matrix, as in the slides near the end of section 5.5.

c) What does $A$ do geometrically? Draw a picture.

Solution.

a) First we compute the characteristic polynomial by expanding cofactors along the third row:

\[
f(\lambda) = \det \begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda) \det \begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix} \\
= (2-\lambda)((4-\lambda)^2 + 9) = (2-\lambda)(\lambda^2 - 8\lambda + 25).
\]

Using the quadratic equation on the second factor, we find the eigenvalues

\[\lambda_1 = 2 \quad \lambda_2 = 4 - 3i \quad \bar{\lambda}_2 = 4 + 3i.\]

Next compute an eigenvector with eigenvalue $\lambda_1 = 2$:

\[
A - 2I = \begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.
\]
The parametric form is \( x = 0, y = z \), so the parametric vector form of the solution is

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = z \begin{pmatrix}
  0 \\
  1 \\
  1
\end{pmatrix}
\]

Now we compute an eigenvector with eigenvalue \( \lambda_2 = 4 - 3i \):

\[
A = (4 - 3i)I = \begin{pmatrix}
  3i & -3 & 3 \\
  3 & 3i & -2 \\
  0 & 0 & 3i - 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  3i & -2 \\
  0 & 3 + 2i \\
  0 & 3i - 2
\end{pmatrix}
\]

\[
\begin{pmatrix}
  1 & i \\
  0 & 0 \\
  0 & 0
\end{pmatrix}
\]

The parametric form of the solution is \( x = -iy, z = 0 \), so the parametric vector form is

\[
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix} = y \begin{pmatrix}
  -i \\
  1 \\
  0
\end{pmatrix}
\]

An eigenvector for the complex conjugate eigenvalue \( \bar{\lambda}_2 = 4 + 3i \) is the complex conjugate eigenvector \( v_2 = \begin{pmatrix}
  1 \\
  0
\end{pmatrix} \).

b) According to the “block-diagonalization” theorem, we have \( A = PCP^{-1} \) where

\[
P = \begin{pmatrix}
  \text{Re} v_2 & \text{Im} v_2 & v_1 \\
  0 & -1 & 0 \\
  1 & 0 & 1 \\
  0 & 0 & 1
\end{pmatrix}
\]

and

\[
C = \begin{pmatrix}
  \text{Re} \lambda_2 & \text{Im} \lambda_2 & 0 \\
  -\text{Im} \lambda_2 & \text{Re} \lambda_2 & 0 \\
  0 & 0 & \lambda_1
\end{pmatrix} = \begin{pmatrix}
  4 & -3 & 0 \\
  3 & 4 & 0 \\
  0 & 0 & 2
\end{pmatrix}.
\]

(I've ordered the eigenvalues in this way to make the picture look nicer in my "z is up" coordinate system.)

c) The matrix \( C \) scales by 2 in the \( z \)-direction, and rotates by \( \arg(-\lambda_2) = \arctan(3/4) \sim .6435 \) radians and scales by \( |\lambda_2| = \sqrt{4^2 + 3^2} = 5 \) in the \( xy \)-directions. The matrix \( A \) does the same thing, with respect to the basis

\[
\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}
\]

of columns of \( P \). [interactive]