Math 1553 Worksheet: Lines and planes in $\mathbb{R}^n$ and §1.1

Solutions

1. Which of the following equations are linear? Justify your answers.

a) $3x_1 + \sqrt{x_2} = 4$

b) $x_1 = x_2 - x_3 + 10x_4$.

c) $\pi x + \ln(13)y + z = \sqrt[3]{2}$

Solution.

a) No. The $\sqrt{x_2}$ term makes it non-linear.

b) Yes.

c) Yes. The $\sqrt[3]{2}$ term is just a constant. Don't be misled by the appearance of the natural logarithm: $\ln(13)$ is just the coefficient for y.

If the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Find all values of h so that the lines $x + hy = -5$ and $2x - 8y = 6$ do *not* intersect.

Solution.

We can use standard algebra, row operations, or geometric intuition.

Using standard algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$x + hy = -5$$
$$2x - 8y = 6.$$

Subtracting twice the first equation from the second equation gives us

$$x + hy = -5$$
$$0 + (-8 - 2h)y = 16.$$

If $-8 - 2h = 0$ (so $h = -4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h = -4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8 - 2h)y = 16 \qquad y = \frac{16}{-8 - 2h} \qquad y = \frac{8}{-4 - h}.$$

We can now substitute this value of y into the first equation to find x:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4 - h} = -5 \qquad x = -5 - \frac{8h}{-4 - h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h = -4}$.

Using row operations: Like the previous technique, let's see what happens if the lines intersect. We put the equations into augmented matrix form and use row operations.

$$\left[\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right] \xrightarrow{R_2=R_2-2R_1} \left[\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right].$$

If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y=16$, so $y=\dfrac{16}{-8-2h}=\dfrac{8}{-4-h}$, and $x=-5-hy=-5-\dfrac{8h}{-4-h}$, so the lines intersect at $\left(-5-\dfrac{8h}{-4-h},\ \dfrac{8}{-4-h}\right)$.

Therefore, our answer is $h=-4$.

Using intuition from geometry in $\mathbb{R}^2$: Two non-identical lines in $\mathbb{R}^2$ intersect if and only if they are not parallel. The second line is $y=\dfrac{1}{4}x-\dfrac{3}{4}$, so its slope is $\dfrac{1}{4}$. If $h \neq 0$, then the first line is $y=-\dfrac{1}{h}x-\dfrac{5}{h}$, so the lines are parallel when $-\dfrac{1}{h}=\dfrac{1}{4}$, which means $h=-4$. You can check that when $h=-4$ the lines aren't identical. (And if $h=0$ then the first line is vertical so it isn't parallel to the second).

3. Consider the following three planes in $\mathbb{R}^3$:

$$\begin{aligned} 2x+4y+4z &= 1 \\ 2x+5y+2z &= -1 \\ y+3z &= 8. \end{aligned}$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

We can isolate z in the third equation using algebra, but it is probably best to do using an augmented matrix and elementary row operations.

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right] \xrightarrow{R_2=R_2-R_1} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right] \xrightarrow{R_3=R_3-R_2} \left[\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right].$$

The last line is $5z=10$, so $z=2$. Since we don't have much practice with row-reduction, we will use substitution to finish.

The second equation is $y-2z=-2$, so $y-2(2)=-2$, thus $y=2$.

The first equation is $2x+4(2)+4(2)=1$, so $2x=-15$, thus $x=-\dfrac{15}{2}$.

We have found that the planes intersect at the point $\left(-\dfrac{15}{2},\ 2,\ 2\right)$.

4. For each of the following, answer true or false. Justify your answer.

a) Every system of linear equations has at least one solution.

b) There is a system of linear equations that has exactly 5 solutions.

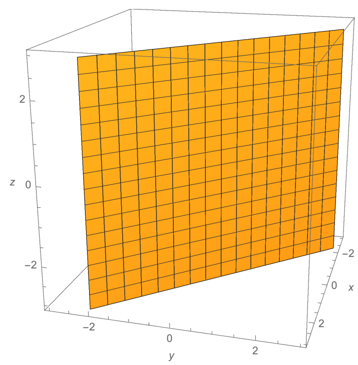
c) If a, b, and c are real numbers, then the equation $ax + by = c$ in $\mathbb{R}^3$ describes a line.

Solution.

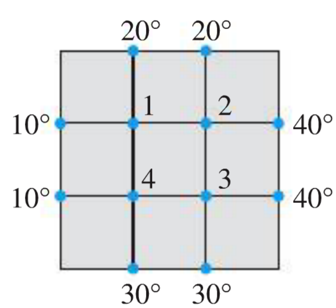
a) False. Some examples from class and this worksheet have no solutions.

b) False. There are only three possibilities: no solutions, exactly one solution, or infinitely many solutions.

c) False. For example, in $\mathbb{R}^3$, the equation $x + y = 1$ describes a vertical plane. We could right the plane in parametric form as $(t,\ 1-t,\ z)$ where t and z vary among all real numbers.



5. The picture below represents the temperatures at four interior nodes of a mesh.



Let $T_1, \ldots, T_4$ be the temperatures at nodes 1 through 4. Suppose that the temperature at each node is the average of the four nearest nodes. For example,

$$T_1 = \frac{10 + 20 + T_2 + T_4}{4}.$$

Write a system of four linear equations whose solution would give the temperatures $T_1, \ldots, T_4$. Next, write an augmented matrix that represents that system of equations.

Solution.

The first equation was given. The others are:

$$T_2 = (T_1 + 20 + 40 + T_3)/4, \quad \text{or} \quad 4T_2 - T_1 - T_3 = 60$$
$$T_3 = (T_4 + T_2 + 40 + 30)/4, \quad \text{or} \quad 4T_3 - T_4 - T_2 = 70$$
$$T_4 = (10 + T_1 + T_3 + 30)/4, \quad \text{or} \quad 4T_4 - T_1 - T_3 = 40$$

To put this in matrix form, we need to put everything in order.

$$\begin{array}{rcrcrcrcr}
4T_1 & - & T_2 & & & - & T_4 & = & 30 \\
-T_1 & + & 4T_2 & - & T_3 & & & = & 60 \\
 & & -T_2 & + & 4T_3 & - & T_4 & = & 70 \\
-T_1 & & & - & T_3 & + & 4T_4 & = & 40
\end{array}$$

This gives the augmented matrix

$$\begin{bmatrix} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{bmatrix}.$$