Math 1553 Worksheet §1.2

Solutions

1. True or false (justify your answer): If a system of linear equations has more variables than equations, it must be consistent.

   Solution.

   False. For example, consider
   $$x + y + z = 0$$
   $$x + y + z = 2.$$

2. **a)** Which of the following matrices are in row echelon form? Which are in reduced row echelon form?

   b) Which entries are the pivots? Which are the pivot columns?

   $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   Solution.

   The first, second, and fourth matrices are in reduced row echelon form; the other is in row echelon form. The pivots are in red; the other entries in the pivot columns are in blue.

3. **a)** Row reduce the following matrices to reduced row echelon form.

   b) If these are augmented matrices for a linear system (with the last column being after the $=$ sign), then which are inconsistent? Which have a *unique* solution?

   Solution.

   $$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 5 & 6 & 7 \\ 6 & 7 & 8 & 9 \end{pmatrix} \xrightarrow{R_2 = R_2 - 4R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 6 & 7 & 8 & 9 \end{pmatrix}$$

   $$\xrightarrow{R_3 = R_3 - 6R_1} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

   $$\xrightarrow{R_2 = R_2 \div -3} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -5 & -10 & -15 \end{pmatrix}$$

   $$\xrightarrow{R_3 = R_3 + 5R_2} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$R_1 = R_1 - 2R_2 \;\rightsquigarrow\; \begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$
\begin{aligned}
x \quad\; - \; z &= -2 \\
y + 2z &= \;\;\, 3 \\
0 &= \;\;\, 0.
\end{aligned}
$$

This system is consistent, but since z is a free variable, it does not have a *unique* solution.

$$\begin{pmatrix} 1 & 3 & 5 & 7 \\ 3 & 5 & 7 & 9 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$R_2 = R_2 - 3R_1 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 5 & 7 & 9 & 1 \end{pmatrix}$$

$$R_3 = R_3 - 5R_1 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & -4 & -8 & -12 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$R_2 = R_2 \div -4 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & -8 & -16 & -34 \end{pmatrix}$$

$$R_3 = R_3 + 8R_2 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & -10 \end{pmatrix}$$

$$R_3 = R_3 \div -10 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = R_1 - 7R_3 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_2 = R_2 - 3R_3 \;\rightsquigarrow\; \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$R_1 = R_1 - 3R_2 \;\rightsquigarrow\; \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$
\begin{aligned}
x \quad\; - \; z &= 0 \\
y + 2z &= 0 \\
0 &= 1,
\end{aligned}
$$

which is inconsistent.

$$\begin{pmatrix} 3 & -4 & 2 & 0 \\ -8 & 12 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix} \quad \xrightarrow{R_2 = R_2 + 3R_1} \quad \begin{pmatrix} 3 & -4 & 2 & 0 \\ 1 & 0 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \longleftrightarrow R_2} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 3 & -4 & 2 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - 3R_1} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ -6 & 8 & -1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 + 6R_1} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & -4 & -4 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 \div -4} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 8 & 11 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 - 8R_2} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_3 \div 3} \quad \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 = R_1 - 2R_3} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - R_3} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

This is the reduced row echelon form. Interpreted as an augmented matrix, it corresponds to the system of linear equations

$$x = 0 \quad y = 0 \quad z = 0,$$

which has a unique solution.

4. What would you have to know about the pivot columns in an augmented matrix in order to know that the corresponding linear system is consistent and has a unique solution?

Solution.

In order for the system to be consistent, the rightmost column must not be a pivot column (otherwise, we would get $0 = 1$). In order for the consistent system to have a unique solution, every column except the rightmost column must be a pivot column (otherwise we would get a free variable and thus infinitely many solutions).

5. Is there a degree-three polynomial $P(x)$ whose graph passes through the points $(-2,6)$, $(-1,4)$, $(1,6)$, and $(2,22)$? If so, how many degree-three polynomials have a graph that through those four points? We will answer this question in steps below.

  a) If $P(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ is a degree-three polynomial passing through the four points listed above, then $P(-2) = 6$, $P(-1) = 4$, $P(1) = 6$, and $P(2) = 22$. Write a system of four equations which we would solve to find a_0, a_1, a_2, and a_3.

  b) Write the augmented matrix to represent this system and put it into reduced row-echelon form. Is the system consistent? How many solutions does it have?

Solution.

  a) We compute

$$
\begin{aligned}
P(-2) &= 6 & \Longrightarrow & & a_0 + a_1 \cdot (-2) + a_2 \cdot (-2)^2 + a_3 \cdot (-2)^3 &= 6, \\
P(-1) &= 4 & \Longrightarrow & & a_0 + a_1 \cdot (-1) + a_2 \cdot (-1)^2 + a_3 \cdot (-1)^3 &= 4, \\
P(1) &= 6 & \Longrightarrow & & a_0 + a_1 \cdot 1 + a_2 \cdot 1^2 + a_3 \cdot 1^3 &= 6, \\
P(2) &= 22 & \Longrightarrow & & a_0 + a_1 \cdot 2 + a_2 \cdot 2^2 + a_3 \cdot 2^3 &= 22.
\end{aligned}
$$

  Simplifying gives us

$$
\begin{aligned}
a_0 - 2a_1 + 4a_2 - 8a_3 &= 6 \\
a_0 - a_1 + a_2 - a_3 &= 4 \\
a_0 + a_1 + a_2 + a_3 &= 6 \\
a_0 + 2a_1 + 4a_2 + 8a_3 &= 22.
\end{aligned}
$$

  b) The corresponding augmented matrix is

$$
\left(\begin{array}{cccc|c}
1 & -2 & 4 & -8 & 6 \\
1 & -1 & 1 & -1 & 4 \\
1 & 1 & 1 & 1 & 6 \\
1 & 2 & 4 & 8 & 22
\end{array}\right)
$$

  We label pivots with boxes as we proceed along. First, we subtract row 1 from each of rows 2, 3, and 4.

$$
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
1 & -1 & 1 & -1 & 4 \\
1 & 1 & 1 & 1 & 6 \\
1 & 2 & 4 & 8 & 22
\end{array}\right)
\rightsquigarrow
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 3 & -3 & 9 & 0 \\
0 & 4 & 0 & 16 & 16
\end{array}\right)
$$

  We now create zeros below the second pivot by subtracting multiples of the second row, then divide by 6 to get

$$
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 0 & \boxed{6} & -12 & 6 \\
0 & 0 & 12 & -12 & 24
\end{array}\right)
\xrightarrow{R_3 = R_3 \div 6}
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 0 & \boxed{1} & -2 & 1 \\
0 & 0 & 12 & -12 & 24
\end{array}\right).
$$

Now we subtract a 12 times row 3 from row 4 and divide by 12:

$$
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 0 & \boxed{1} & -2 & 1 \\
0 & 0 & {\color{red}0} & \boxed{12} & 12
\end{array}\right)
\quad \xrightarrow{R_4 = R_4 \div 12} \quad
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 0 & \boxed{1} & -2 & 1 \\
0 & 0 & 0 & \boxed{1} & {\color{red}1}
\end{array}\right).
$$

At this point we can actually use back-substitution to solve: the last row says $a_3 = 1$, then plugging in $a_3 = 1$ in the third row gives us $a_2 = 3$, etc. However, for the sake of practice with reduced echelon form, let's keep row-reducing.

From right to left, we create zeros above the pivots in pivot columns by subtracting multiples of the pivot columns.

$$
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & -8 & 6 \\
0 & \boxed{1} & -3 & 7 & -2 \\
0 & 0 & \boxed{1} & -2 & 1 \\
0 & 0 & 0 & \boxed{1} & 1
\end{array}\right)
\quad \rightsquigarrow \quad
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & 4 & {\color{red}0} & 14 \\
0 & \boxed{1} & -3 & {\color{red}0} & -9 \\
0 & 0 & \boxed{1} & {\color{red}0} & 3 \\
0 & 0 & 0 & \boxed{1} & 1
\end{array}\right)
$$

$$
\rightsquigarrow \quad
\left(\begin{array}{cccc|c}
\boxed{1} & -2 & {\color{red}0} & 0 & 2 \\
0 & \boxed{1} & {\color{red}0} & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 3 \\
0 & 0 & 0 & \boxed{1} & 1
\end{array}\right)
$$

$$
\rightsquigarrow \quad
\left(\begin{array}{cccc|c}
\boxed{1} & {\color{red}0} & 0 & 0 & 2 \\
0 & \boxed{1} & 0 & 0 & 0 \\
0 & 0 & \boxed{1} & 0 & 3 \\
0 & 0 & 0 & \boxed{1} & 1
\end{array}\right)
$$

So $a_0 = 2$, $a_1 = 0$, $a_2 = 3$, and $a_3 = 1$. In other words,

$$
\boxed{P(x) = 2 + 3x^2 + x^3}.
$$

Therefore, there is exactly one third-degree polynomial satisfying the conditions of the problem. (You should check that, in fact, we have $P(-2) = 6$, $P(-1) = 4$, etc.)