Math 1553 Worksheet §1.4

Solutions

For problems 1, 2, and 3 below: The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3. Write a matrix equation $Ax = b$ to relate your final grades to your scores.

Solution.

In the above grading schemes, you would receive the following final grades:

$$\begin{aligned} &\text{Scheme 1:} \quad 0.1x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 = b_1 \\ &\text{Scheme 2:} \quad 0.1x_1 + 0.1x_2 + 0.4x_3 + 0.4x_4 = b_2 \\ &\text{Scheme 3:} \quad 0.1x_1 + 0.1x_2 + 0.6x_3 + 0.2x_4 = b_3 \end{aligned}$$

This is the same as the matrix equation

$$(*) \qquad \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

2. Suppose that you end up with averages of 90% on the homework, 90% on quizzes, 85% on midterms, and a 95% score on the final exam. Use Problem 1 to determine which grading scheme leaves you with the highest overall course grade.

Solution.

According to equation (*) above, your final grades would be

$$\begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} .90 \\ .90 \\ .85 \\ .95 \end{pmatrix} = \begin{pmatrix} .89 \\ .90 \\ .88 \end{pmatrix}.$$

Hence the second grading scheme gives you the best final grade.

3. **a)** Keeping $b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ as a general vector, write the augmented matrix $(A \mid b)$.

b) Row reduce this matrix until you reach row echelon form.

c) Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?

d) The answer to (c) also defines the span of the columns of A. Describe the span geometrically.

e) Solve the equation in (c) for b_1. Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?

Solution.

a) $\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right)$

b) Here is the row reduction:

$$\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right) \xrightarrow{\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - R_1}} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2 - b_1 \\ 0 & 0 & 0.1 & -0.1 & b_3 - b_1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + R_2} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

$$\xrightarrow{\substack{R_1 = R_1 \times 10 \\ R_2 = R_2 \times (-10)}} \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 10b_1 \\ 0 & 0 & 1 & -1 & 10b_1 - 10b_2 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 5R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 8 & -40b_1 + 50b_2 \\ 0 & 0 & 1 & -1 & 10b_1 - 10b_2 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

c) The last row in the row-reduced matrix translates into $0 = b_2 + b_3 - 2b_1$. Hence the system of equations is inconsistent unless $\boxed{b_2 + b_3 - 2b_1 = 0}$.

d) This is the 2-plane in $\mathbf{R}^3$ given by $-2b_1 + b_2 + b_3 = 0$.

e) Rearranging, this is the set of points (b_1, b_2, b_3) where $b_1 = \frac{1}{2}(b_2 + b_3)$, i.e., where b_1 is the average of b_2 and b_3. Hence it is impossible for b_1 to be larger than both b_2 and b_3.

You should argue for the second grading scheme if your final grade was higher than your midterm grade; otherwise you should argue for the third.

4. True or false. If the statement is ever false, answer false. Justify your answer.

a) A matrix equation $Ax = b$ is consistent if A has a pivot in every column.

b) If $Ax = b$ is inconsistent, then b is not in the span of the columns of A.

c) If A is a 5×4 matrix, then the equation $Ax = b$ must be inconsistent for some b in $\mathbf{R}^5$.

Solution.

a) False. For example, the system $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ has no solution, even though the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has a pivot in every column. However, the system is guaranteed to be consistent if A has a pivot in every **row**.

b) True. The question of whether $Ax = b$ is consistent or inconsistent is just the question of whether b is, or is not, in the span of the columns of A.

c) True. If A is a 5×4 matrix, then A can have at most 4 pivots (since no row or column can have more than 1 pivot). But A has 5 rows, so this means A cannot have a pivot in each row, and therefore $Ax = b$ must be inconsistent for at least one b in $\mathbf{R}^5$.

5. Find the solution sets of $x_1 - 3x_2 + 5x_3 = 0$ and $x_1 - 3x_2 + 5x_3 = 3$. How do they compare geometrically? You may want to sketch the two planes to see the picture.

Solution.

The equation $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

The equation $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

Geometrically, we see that to get to a solution for the second equation, we simply translate a solution to the first equation by $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$.

In section 1.5, we will see that these solutions can be written in a particular parametric form. The general solution to the equation $x_1 - 3x_2 + 5x_3 = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The general solution to the equation $x_1 - 3x_2 + 5x_3 = 0$ is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.