Math 1553 Worksheet §2.1, 2.2, 2.3

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined?

a) $A - B$

b) AB

c) $A^T B$

d) $B^T A$

e) A^2

Solution.

Only (c) and (d).
$A - B$ is nonsense. In order for $A - B$ to be defined, A and B need to have the same number of rows and same number of columns as each other.
AB is undefined since the number of columns of A does not equal the number of rows of B.
A^T is 5×3 and B is 3×2, so $A^T B$ is a 5×2 matrix.
B^T is 2×3 and A is 3×5, so $B^T A$ is a 2×5 matrix.
A^2 is nonsense (can't do 3×5 times 3×5).

2. Find all matrices B that satisfy

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}.$$

Solution.

B must have two rows and two columns for the above to compute, so $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.
We calculate

$$\begin{pmatrix} 1 & -3 \\ -3 & 5 \end{pmatrix} B = \begin{bmatrix} a - 3c & b - 3d \\ -3a + 5c & -3b + 5d \end{bmatrix}.$$

Setting this equal to $\begin{pmatrix} -3 & -11 \\ 1 & 17 \end{pmatrix}$ gives us

$$\begin{gathered} a - 3c = -3, \\ -3a + 5c = 1, \\ \text{(solving gives us } a = 3,\ c = 2) \\ b - 3d = -11, \\ -3b + 5d = 17. \\ \text{(solving gives us } b = 1,\ d = 4). \end{gathered}$$

Therefore, $B = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

3. **a)** If the columns of an $n \times n$ matrix Z are linearly independent, is Z necessarily invertible? Justify your answer.

b) Solve $AB = BC$ for A, assuming A, B, C are $n \times n$ matrices and B is invertible. Be careful!

Solution.

a) Yes. The transformation $x \to Zx$ is one-to-one since the columns of Z are linearly independent. Thus Z has a pivot in all n columns, so Z has n pivots. Since Z also has n rows, this means that Z has a pivot in every row, so $x \to Zx$ is onto. Therefore, Z is invertible.

Alternatively, since Z is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem (2.3) in 2.3 says that Z is invertible.

b)

$$AB = BC \qquad AB(B^{-1}) = BC(B^{-1}) \qquad AI_n = BCB^{-1} \qquad \boxed{A = BCB^{-1}}.$$

It is very important that we multiplied by B^{-1} on the same side in each equation, since matrix multiplication generally is not commutative.

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is an $m \times n$ matrix and B is an $n \times p$ matrix, then each column of AB is a linear combination of the columns of A.

b) If A and B are $n \times n$ and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

c) If A^T is not invertible, then A is not invertible.

d) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

e) If A and B are invertible $n \times n$ matrices, then $A + B$ is invertible and $(A+B)^{-1} = A^{-1} + B^{-1}$.

f) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, then $Ax = 0$ has a unique solution.

Solution.

a) True. If we let $v_1, \ldots, v_p$ be the columns of B, then $AB = \begin{pmatrix} Av_1 & Av_2 & \cdots & Av_p \end{pmatrix}$, where Av_i is in the column span of A for every i (this is part of the definition of matrix multiplication of vectors).

b) False. $(AB)^{-1} = B^{-1}A^{-1}$.

c) True. If there is a matrix A so that A^T is not invertible but A *is* invertible, then from our notes in 2.2 it would follow that A^T is invertible in the first place!

Alternatively, this problem could be quoted as part of the Invertible Matrix Theorem in 2.3.

d) True. The first part says $x \to Ax$ is onto. Since A is $n \times n$, this is the same as saying A is invertible, so $x \to Ax$ is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

e) False. $A + B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A + B = 0$ which is not invertible. Even in the case when $A + B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

f) True. Since AB is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that AB is invertible. Note A is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since A is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.

5. Suppose A is an invertible 3×3 matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \quad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \quad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A.

Solution.

The columns of A^{-1} are

$$(A^{-1}e_1 \;\; A^{-1}e_2 \;\; A^{-1}e_3), \qquad \text{so} \qquad A^{-1} = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get A, we just find $(A^{-1})^{-1}$. Row-reducing $(A^{-1} \mid I)$ eventually gives us

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right), \quad \text{so} \quad A = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$