Announcements: August 30

- Office Hours today 3-4, Skiles 234
- Qianli Office Hours today 1-2, Skiles 153
- Kemi Office Hours Thursday 9:30-10:30, Skiles 230
- Martin's Office Hours Friday 2-3, Skiles 230
- WebWorK due Friday (only this week)
- Quiz in recitation on Friday (covers material up to Monday's class)
- Join our Piazza group: 1553 E1 through E5
- Set up your device so that you can quickly access Piazza during class

Section 1.2

Row Reduction and Echelon Forms

Row Reduction and Echelon Forms

A matrix is in row echelon form if

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

$$\begin{pmatrix} \boxed{\star} & \star & \star & \star & \star \\ 0 & \boxed{\star} & \star & \star & \star \\ 0 & 0 & 0 & \boxed{\star} & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is easy to solve using back substitution.

The pivot positions are the leading (nonzero) entries in each row.

Reduced Row Echelon Form

A system is in reduced row echelon form if also:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

For example:

$$\begin{pmatrix} 1 & 0 & \star & 0 & \star \\ 0 & 1 & \star & 0 & \star \\ 0 & 0 & 0 & 1 & \star \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This system is even easier to solve.

Can every matrix be put in reduced row echelon form?

Reduced Row Echelon Form

Poll

Which are in reduced row echelon form?

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 8 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 17 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

REF:

1. all zero rows are at the bottom,
2. each leading (nonzero) entry of a row is to the right of the leading entry of the row above, and
3. below a leading entry of a row all entries are zero.

RREF:

4. the leading entry in each nonzero row is 1
5. each leading entry of a row is the only nonzero entry in its column

Row Reduction

Theorem. Each matrix is row equivalent to one and only one matrix in reduced row echelon form.

We'll give an algorithm. That shows a matrix is equivalent to at least one matrix in reduced row echelon form.

Row Reduction Algorithm

To find row echelon form:

Step 1 Swap rows so a leftmost nonzero entry is in 1st row (if needed)

Step 2 Scale 1st row so that its leading entry is equal to 1

Step 3 Use row replacement so all entries below this 1 (or, pivot) are 0

Then cover the first row and repeat the three steps.

To then find reduced row echelon form:

- Use row replacement so that all entries above the pivots are 0.

Examples.

$$\left(\begin{array}{ccc|c} 0 & 7 & -4 & 2 \\ 2 & 4 & 6 & 12 \\ 3 & 1 & -1 & -2 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right)$$

Solutions of Linear Systems

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 2 \end{array}\right)$$

What are the solutions?

Solutions of Linear Systems: Consistency

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Solutions of Linear Systems: Free Variables I

We want to go from reduced row echelon forms to solutions of linear systems.

Solve the linear system associated to:

$$\left(\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 2 & 1 \end{array}\right)$$

This represents two equations:

$$\begin{aligned} x_1 + 5x_3 &= 0 \\ x_2 + 2x_3 &= 1 \end{aligned}$$

Solutions of Linear Systems: Free Variables II

Solve the linear system associated to:

$$\left(\begin{array}{cccc|c} 1 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

Summary

There are *three possibilities* for the reduced row echelon form of the augmented matrix of a linear system.

1. The last column is a pivot column.
   $\rightsquigarrow$ the system is *inconsistent*.

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$$

2. Every column except the last column is a pivot column.
   $\rightsquigarrow$ the system has a *unique solution*.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & \star \\ 0 & 1 & 0 & \star \\ 0 & 0 & 1 & \star \end{array}\right)$$

3. The last column is not a pivot column, and some other column isn't either.
   $\rightsquigarrow$ the system has *infinitely many* solutions; free variables correspond to columns without pivots.

$$\left(\begin{array}{cccc|c} 1 & \star & 0 & \star & \star \\ 0 & 0 & 1 & \star & \star \end{array}\right)$$

Example with a parameter

For which values of h does the following system have a solution? For which values of h does it have a unique solution?

$$\begin{aligned} x + y &= 1 \\ 2x + 2y &= h \end{aligned}$$

Poll

Poll

A linear system has 4 variables and 3 equations. What are the possible solution sets?

1. nothing
2. point
3. line
4. plane
5. 3-dimensional plane
6. 4-dimensional plane