Announcements: Sep 18

- Midterm 1 on Friday
- WebWork due Wednesday
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- WebWorK due Wednesday
- Quiz in recitation on Friday (covers material from last week, Sec 1.3)

Learning goals

- When does a set of vectors span $\mathbb{R}^m$?
- Solutions to $Ax = b$ form a plane, parallel to the solutions to $Ax = 0$
- How to write the parametric form for a solution

From last time: Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is...
3. A has a pivot in each...

Why?

Section 1.5

Solution Sets of Linear Systems

Homogeneous systems

Homogeneous systems $\longleftrightarrow$ matrix equations $Ax = 0$.

Homogenous systems are always consistent.

When does $Ax = 0$ have a nonzero solution?

Homogeneous systems
Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Homogeneous systems

Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

Homogeneous systems
Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Homogeneous systems
Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & -8 & -7 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Homogeneous systems

Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Homogeneous systems
Example

Describe geometrically the solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Dimension and Span of Homogeous Systems

- If $v_1, \ldots, v_k$ are solutions to $Ax = 0$, then so is...

  Why?

- $\rightsquigarrow$ set of solutions to $Ax = 0$ is...

Parametric Forms

Say free variables for $Ax = 0$ are $x_k, \dots, x_n$.

Then the solutions to $Ax = 0$ can be written as

$$x_k v_k + x_{k+1} v_{k+2} + \cdots + x_n v_n$$

for some $v_k, \dots, v_n$ (in other words, as a span!).

This is the *parametric form* of the solutions.

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 0 \\ -2 & -3 & 4 & 5 & 0 \\ 2 & 4 & 0 & -2 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & 0 \\ 0 & 1 & 4 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Note: don't really need the last column!

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\rightsquigarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

Parametric Forms for Solutions

Homogeneous case

Find the parametric solution to $Ax = 0$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

Variables, equations, and dimension

Poll

For $b \neq 0$, the solutions to $Ax = b$ are...

1. always a span
2. sometimes a span
3. never a span

Nonhomogeneous Systems

Suppose $Ax = b$, and $b \neq 0$.

As before, we can find the parametric solution in terms of free variables.

What is the difference?

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (5, -10)$ where

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (3, 2, 6)$ where

$$A = \begin{pmatrix} 1 & 2 & 0 & -1 \\ -2 & -3 & 4 & 5 \\ 2 & 4 & 0 & -2 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & -1 & 3 \\ -2 & -3 & 4 & 5 & 2 \\ 2 & 4 & 0 & -2 & 6 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -8 & -7 & -13 \\ 0 & 1 & 4 & 3 & 8 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (4, 2, 4)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 4 \\ 1 & 0 & 0 & 1 & 2 \\ 1 & 2 & 1 & 0 & 4 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccr|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

Parametric Forms for Solutions

Nonhomogeneous case

Find the parametric solution to $Ax = (9)$ where

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 9 \end{array} \right)$$

Homogeneous vs. Nonhomogeneous Systems

Key realization. Set of solutions to $Ax = b$ obtained by taking one solution and adding all possible solutions to $Ax = 0$.

$$Ax = 0 \text{ solutions} \rightsquigarrow Ax = b \text{ solutions}$$

$$x_k v_k + \cdots + x_n v_n \rightsquigarrow$$

So: set of solutions to $Ax = b$ is to the set of solutions to $Ax = 0$.

So by understanding $Ax = 0$ we gain understanding of $Ax = b$ for all b. This gives structure to the set of equations $Ax = b$ for all b.

Homogeneous vs. Nonhomogeneous Systems

Varying b

What are the solutions to $Ax = b$ for various b where

$$A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}?$$