Announcements: Sep 18

- No quiz Friday
- WebWork due Friday
- Midsemester reflections on Canvas
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Midterm 2 October 20

Learning goals

Sections 1.8-1.9

- Learn to think of matrices as functions, called matrix transformations
- Understand what certain specific matrices do to $\mathbb{R}^n$
- Understand the definition of a linear transformation
- Linear transformations are the same as matrix transformations
- Determine when a linear transformation is one-to-one, onto

Sections 1.8-1.9

Linear Transformations

Examples in $\mathbb{R}^2$

What does each matrix do to $\mathbb{R}^2$?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

Examples in $\mathbb{R}^2$

What does each matrix do to $\mathbb{R}^2$?

Hint: if you can't see it all at once, see what happens to the x- and y-axes.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Examples in $\mathbb{R}^3$

What does each matrix do to $\mathbb{R}^3$?

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Section 1.8-1.9

Linear Transformations II

Linear transformations are matrix transformations

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if

- $T(u+v) = T(u) + T(v)$ for all u, v in $\mathbb{R}^n$.
- $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and c in $\mathbb{R}$.

Main point: if we know $T(e_1), \ldots, T(e_n)$, then we know every $T(v)$.

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \end{aligned}$$

Theorem. A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it is a matrix transformation.

Linear transformations are matrix transformations

Theorem. A function $\mathbb{R}^n \to \mathbb{R}^m$ is linear if and only if it is a matrix transformation.

The matrix for a linear transformation is called the standard matrix.

Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \to \mathbb{R}^3$ is the function given by:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ y \\ x-y \end{pmatrix}$$

What is the standard matrix for T?

In fact, a function $\mathbb{R}^n \to \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables).

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the x-direction and 3 in the y-direction, and then reflects over the line $y = x$.

Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that projects onto the y-axis and then rotates counterclockwise by $\pi/2$.

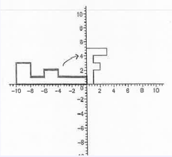
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^3$ that reflects through the xy-plane and then projects onto the yz-plane.

Discussion

Discussion Question

Find a matrix that does this.



One-to-one

$T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each b in $\mathbb{R}^m$ is the image of at most one v in $\mathbb{R}^n$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A. Then the following are all equivalent:

- T is one-to-one
- the columns of A are...
- $Ax = 0$ has...
- A has a pivot...
- the range has dimension...

What can we say about the relative sizes of m and n if T is one-to-one?

Draw a picture of the range of a one-to-one mapping $\mathbb{R} \to \mathbb{R}^3$.

Onto

$T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of T is $\mathbb{R}^m$, that is, each b in $\mathbb{R}^m$ is the image of at least one v in $\mathbb{R}^m$.

Theorem. Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix A. Then the following are all equivalent:

- T is onto
- the columns of A...
- A has a pivot...
- $Ax = b$ is consistent...

What can we say about the relative sizes of m and n if T is onto?

Give an example of an onto mapping $\mathbb{R}^3 \to \mathbb{R}$.

One-to-one and Onto

Do the following give linear transformations that are one-to-one? onto?

$$\begin{pmatrix} 1 & 0 & 7 \\ 0 & 1 & 2 \\ 0 & 0 & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$