Announcements: Oct 2

- WebWork 1.8 & 1.9 due Wednesday
- Quiz on 1.7, 1.8, & 1.9 on Friday
- Midsemester reflections on Canvas
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

- Midterm 2 October 20
Learning goals

Sections 1.8-1.9

- How to find the matrix for a linear transformation
- Determine when a linear transformation is one-to-one, onto

Section 2.1

- How to multiply matrices
- Understand composition of linear transformations
Section 1.8-1.9
Linear Transformations II
Linear transformations are matrix transformations

Suppose $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ is the function given by:

$$T \left( \begin{pmatrix} x \\ y \end{pmatrix} \right) = \begin{pmatrix} x + y \\ y \\ x - y \end{pmatrix}$$

What is the standard matrix for $T$?

In fact, a function $\mathbb{R}^n \rightarrow \mathbb{R}^m$ is linear exactly when the coordinates are linear (linear combinations of the variables).
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of $\mathbb{R}^2$ that stretches by 2 in the $x$-direction and 3 in the $y$-direction, and then reflects over the line $y = x$. 
**One-to-one**

$T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is one-to-one if each $b$ in $\mathbb{R}^m$ is the image of at most one $v$ in $\mathbb{R}^n$.

**Theorem.** Suppose $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a linear transformation with standard matrix $A$. Then the following are all equivalent:

- $T$ is one-to-one
- the columns of $A$ are linearly independent
- $Ax = 0$ has only the trivial solution
- $A$ has a pivot in each column
- the range has dimension $n$

What can we say about the relative sizes of $m$ and $n$ if $T$ is one-to-one?

Draw a picture of the range of a one-to-one mapping $\mathbb{R} \rightarrow \mathbb{R}^3$. 
Onto

\( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is onto if the range of \( T \) is \( \mathbb{R}^m \), that is, each \( b \) in \( \mathbb{R}^m \) is the image of at least one \( v \) in \( \mathbb{R}^m \).

**Theorem.** Suppose \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a linear transformation with standard matrix \( A \). Then the following are all equivalent:

- \( T \) is onto
- the columns of \( A \) span \( \mathbb{R}^m \)
- \( A \) has a pivot in each row
- \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^m \).

What can we say about the relative sizes of \( m \) and \( n \) if \( T \) is onto?

Give an example of an onto mapping \( \mathbb{R}^3 \rightarrow \mathbb{R} \).
Section 2.1
Matrix Operations
**Terminology**

Suppose $A$ is an $m \times n$ matrix.

$a_{ij}$ or $A_{ij}$ is the $ij$th entry

$a_{ii}$ are diagonal entries

**diagonal matrix**: all non-diagonal entries are 0

**identity matrix**: diagonal matrix with 1's on the diagonal

**zero matrix**: all entries are 0

the transpose of $A$ is denoted $A^T$ and has $ij$ entry $a_{ji}$
Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

\[ A + B = \]

\[ (A + B) + C = \]

\[ r(A + B) = \]

\[ (r + s)A = \]

\[ (rs)A = \]

\[ A + 0 = \]
Matrix Multiplication

If $A$ is $m \times n$ and $B$ is $n \times p$, then $AB$ is $m \times p$ and

$$(AB)_{ij} = r_i \cdot b_j$$

where $r_i$ is the $i$th row of $A$, and $b_j$ is the $j$th column of $B$.

Or:
Matrix Multiplication and Linear Transformations

The composition $T \circ U$ means: do $U$ then do $T$

Example. $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

Question. What is the standard matrix for $T \circ U$?

Fact. The matrix for a composition of linear transformations is the product of the standard matrices.

Check by plugging in the $e_i$ to $T \circ U$ and to the corresponding product.
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No
Properties of Matrix Multiplication

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $r(AB) = (rA)B = A(rB)$
- $(AB)^T = B^T A^T$
- $I_mA = A = AI_n$, where $I_m$ is the $m \times m$ identity matrix.

Multiplication is associative because function composition is.

Warning!

- $AB$ is not always equal to $BA$
- $AB = AC$ does not mean that $B = C$
- $AB = 0$ does not mean that $A$ or $B$ is 0