Announcements: Oct 4

- WebWork 1.8 & 1.9 due Wednesday
- Quiz on 1.7, 1.8, & 1.9 on Friday
- Thanks for doing Midsemester reflections - more soon!
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Midterm 2 October 20
Learning goals

Section 2.2
- Definition and basic properties of matrix inverses
- Computing the inverse of a matrix
- Define elementary matrices and explain the relationship to row operations and inverses

Section 2.3
- Learn the invertible matrix theorem and how to apply it.
Section 2.2
The Inverse of a Matrix
Inverses

A = $n \times n$ matrix.

A is invertible if there is a matrix B with

\[ AB = BA = I_n \]

B is called the inverse of A and is written $A^{-1}$

Example:

\[
\begin{pmatrix}
2 & 1 \\
1 & 1 \\
\end{pmatrix}^{-1} = \begin{pmatrix}
1 & -1 \\
-1 & 2 \\
\end{pmatrix}
\]
Inverses

Can you guess the inverse of \[
\begin{pmatrix}
1 & 1 \\
0 & 1
\end{pmatrix}
\]?

Find a $2 \times 2$ matrix that is not invertible.
The 2 × 2 Case

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( \det(A) = ad - bc \) is the determinant of \( A \).

*Fact.* If \( \det(A) \neq 0 \) then \( A \) is invertible and \( A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \).

If \( \det(A) = 0 \) then \( A \) is not invertible.

*Example.* \( \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \)
Solving Linear Systems via Inverses

Fact. If $A$ is invertible, then $Ax = b$ has exactly one solution...

Solve

\[\begin{align*}
2x + 3y + 2z &= 1 \\
x + 3z &= 1 \\
2x + 2y + 3z &= 1
\end{align*}\]

Using

\[
\begin{pmatrix}
2 & 3 & 2 \\
1 & 0 & 3 \\
2 & 2 & 3
\end{pmatrix}
^{-1} = \begin{pmatrix}
-6 & -5 & 9 \\
3 & 2 & -4 \\
2 & 2 & -3
\end{pmatrix}
\]
Solving Linear Systems via Inverses

What if we change \( b \)?

\[
\begin{align*}
2x + 3y + 2z &= 1 \\
x + 3z &= 0 \\
2x + 2y + 3z &= 1
\end{align*}
\]

Using

\[
\begin{pmatrix}
2 & 3 & 2 \\
1 & 0 & 3 \\
2 & 2 & 3
\end{pmatrix}
^{-1}
= 
\begin{pmatrix}
-6 & -5 & 9 \\
3 & 2 & -4 \\
2 & 2 & -3
\end{pmatrix}
\]
Some Facts

Say that $A$ and $B$ are invertible $n \times n$ matrices.

- $A^{-1}$ is invertible and $(A^{-1})^{-1} = A$
- $AB$ is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- $A^T$ is invertible and $(A^T)^{-1} = (A^{-1})^T$

What is $(ABC)^{-1}$?
An Algorithm for Finding $A^{-1}$

Suppose $A = n \times n$ matrix.

- Row reduce $(A \mid I_n)$
- If reduction has form $(I_n \mid B)$ then $A$ is invertible and $B = A^{-1}$.
- Otherwise, $A$ is not invertible.

Exercise. Find

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{pmatrix} \rightsquigarrow$$
Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

\[Ax_1 = e_1\]
\[Ax_2 = e_2\]

and so on. But the columns of \( A^{-1} \) are \( A^{-1}e_i \), which is \( x_i \).

There is another explanation, which uses elementary matrices.
Elementary matrices

An elementary matrix, $E$, is one that differs by $I_n$ by one row operation.

If $E$ is an elementary matrix for some row operation, then $EA$ differs from $A$ by same row operation.

Why? Check for each type.

**Fact.** Elementary matrices are invertible.
Elementary matrices

Two matrices are row equivalent if they differ by row operations.

**Observation.** An $n \times n$ matrix $A$ is invertible exactly when it is row equivalent to $I_n$. In this case, the sequence of row operations taking $A$ to $I_n$ also takes $I_n$ to $A^{-1}$. This gives us a second explanation of the algorithm.
Section 2.3
Characterizations of Invertible Matrices
The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

a) $A$ is invertible
b) $A$ is row equivalent to $I_n$
c) $A$ has $n$ pivots
d) $Ax = 0$ has only 0 solution
e) columns of $A$ are linearly independent
f) $T$ is one-to-one
g) $Ax = b$ is consistent for all $b$ in $\mathbb{R}^n$
h) columns of $A$ span $\mathbb{R}^n$
i) $T$ is onto
j) $A^T$ is invertible

Why are these statements all equivalent?
The Invertible Matrix Theorem

Say \( A = n \times n \) matrix and \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the associated linear transformation. The following are equivalent.

a) \( A \) is invertible
b) \( A \) is row equivalent to \( I_n \)
c) \( A \) has \( n \) pivots
d) \( Ax = 0 \) has only 0 solution
e) columns of \( A \) are linearly independent
f) \( T \) is one-to-one
g) \( Ax = b \) is consistent for all \( b \) in \( \mathbb{R}^n \)
h) columns of \( A \) span \( \mathbb{R}^n \)
i) \( T \) is onto
j) \( A^T \) is invertible

Why are these statement all equivalent?
The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.
Example

Determine whether $A$ is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn’t necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

There are three pivot positions, so $A$ is invertible by the IMT (statement c).
The Invertible Matrix Theorem

Poll
Which are true?

m) If $A$ is invertible then the rows of $A$ span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all $b$ in $\mathbb{R}^n$ then $A$ is row equivalent to the identity.

o) If $A$ is invertible then $A^2$ is invertible

p) If $A^2$ is invertible then $A$ is invertible
Identity transformation

The identity linear transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the one where $T(v) = v$ for all $v$.

What is the standard matrix?
Invertible Functions

A function $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is invertible if there is a function $U : \mathbb{R}^n \rightarrow \mathbb{R}^n$, so

$$T \circ U = \text{identity}$$

That is,

$$T \circ U(v) = v \text{ for all } v \in \mathbb{R}^n$$

Fact. Suppose $A = n \times n$ matrix and $T$ is the matrix transformation. Then $T$ is invertible as a function if and only if $A$ is invertible. And in this case, the standard matrix for $T^{-1}$ is $A^{-1}$.

Example. Counterclockwise rotation by $\pi/4$. 
Which of the following linear transformations of $\mathbb{R}^3$ have invertible standard matrices?

- projection to $xy$-plane
- rotation about $z$-axis by $\pi$
- reflection through $xy$-plane