Announcements: Oct 4

- WebWork 1.8 & 1.9 due Wednesday
- Quiz on 1.7, 1.8, & 1.9 on Friday
- Thanks for doing Midsemester reflections - more soon!
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Midterm 2 October 20

Learning goals

Section 2.2

- Definition and basic properties of matrix inverses
- Computing the inverse of a matrix
- Define elementary matrices and explain the relationship to row operations and inverses

Section 2.3

- Learn the invertible matrix theorem and how to apply it.

Section 2.2

The Inverse of a Matrix

Inverses

$A = n \times n$ matrix.

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Inverses

Can you guess the inverse of $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$?

Find a 2×2 matrix that is not invertible.

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} =$

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution...

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- A^T is invertible and $(A^T)^{-1} = (A^{-1})^T$

What is $(ABC)^{-1}$?

An Algorithm for Finding A^{-1}

Suppose $A = n \times n$ matrix.

- Row reduce $(A \,|\, I_n)$
- If reduction has form $(I_n \,|\, B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Exercise. Find $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow$$

Why Does This Work?

First answer: we can think of the algorithm as simultanenously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

There is another explanation, which uses elementary matrices.

Elementary matrices

An elementary matrix, E, is one that differs by I_n by one row operation.

If E is an elementary matrix for some row operation, then EA differs from A by same row operation.

Why? Check for each type.

Fact. Elementary matrices are invertible.

Elementary matrices

Two matrices are row equivalent if they differ by row operations.

Observation. An $n \times n$ matrix A is invertible exactly when it is row equivalent to I_n. In this case, the sequence of row operations taking A to I_n also takes I_n to A^{-1}. This gives us a second explanation of the algorithm.

Section 2.3

Characterizations of Invertible Matrices

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

a) A is invertible
b) A is row equivalent to I_n
c) A has n pivots
d) $Ax = 0$ has only 0 solution
e) columns of A are linearly independent
f) T is one-to-one
g) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
h) columns of A span $\mathbb{R}^n$
i) T is onto
j) A^T is invertible

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

a) A is invertible
b) A is row equivalent to I_n
c) A has n pivots
d) $Ax = 0$ has only 0 solution
e) columns of A are linearly independent
f) T is one-to-one
g) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
h) columns of A span $\mathbb{R}^n$
i) T is onto
j) A^T is invertible

Why are these statement all equivalent?

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \backsim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \backsim \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

The Invertible Matrix Theorem

Poll

Which are true?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

o) If A is invertible then A^2 is invertible

p) If A^2 is invertible then A is invertible

Identity transformation

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is the one where $T(v) = v$ for all v.

What is the standard matrix?

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = \text{identity}$$

That is,

$$T \circ U(v) = v \text{ for all } v \in \mathbb{R}^n$$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the standard matrix for T^{-1} is A^{-1}.

Example. Counterclockwise rotation by $\pi/4$.

Linear Transformations and Inverses

Poll

Which of the following linear transformations of $\mathbb{R}^3$ have invertible standard matrices?

- projection to xy-plane
- rotation about z-axis by π
- reflection through xy-plane