Announcements: Oct 11

- WebWork 2.1, 2.2, & 2.3 due tonight
- Quiz on 2.1, 2.2, & 2.3 on Friday
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Midterm 2 October 20 on 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, & 2.9

Section 2.3

Characterizations of Invertible Matrices

Learning goals

- Learn the invertible matrix theorem and how to apply it.

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

a) A is invertible
b) A is row equivalent to I_n
c) A has n pivots
d) $Ax = 0$ has only 0 solution
e) columns of A are linearly independent
f) T is one-to-one
g) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
h) columns of A span $\mathbb{R}^n$
i) T is onto
j) A^T is invertible

There are two kinds of square matrices, invertible and non-invertible matrices. For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

The Invertible Matrix Theorem

Poll

Which are true?

m) If A is invertible then the rows of A span $\mathbb{R}^n$

n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.

o) If A and B are invertible then AB is invertible

p) If A^2 is invertible then A is invertible

Identity transformation

The identity linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is the one where $T(v) = v$ for all v.

What is the matrix for T?

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = \text{identity}$$

That is,

$$T \circ U(v) = v \text{ for all } v \in \mathbb{R}^n$$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the matrix for T^{-1} is A^{-1}.

Example. Counterclockwise rotation by $\pi/4$.

Linear Transformations and Inverses

Poll

Which of the following linear transformations of $\mathbb{R}^3$ have invertible matrices?

- projection to xy-plane
- rotation about z-axis by π
- reflection through xy-plane

Section 2.8

Subspaces of $\mathbb{R}^n$

Learning goals

- Definition of subspace
- Examples and non-examples of subspaces
- Subspaces are the same as spans
- Bases for subspaces
- Two important subspaces for a matrix: $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Subspaces

A subspace of $\mathbb{R}^n$ is a subset V with:

1. The zero vector is in V.
2. If u and v are in V, then $u + v$ is also in V.
3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.

Which are subspaces?

1. the unit circle in $\mathbb{R}^2$
2. the point $(1, 2, 3)$ in $\mathbb{R}^3$
3. the xy-plane in $\mathbb{R}^3$
4. the xy-plane together with the z-axis in $\mathbb{R}^3$

Spans are subspaces

Fact. Any $\mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace.

Which are subspaces?

1. $$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ in } \mathbb{R}^4 \,\middle|\, a+b+c+d=0 \right\}$$

2. $$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ in } \mathbb{R}^4 \,\middle|\, a+b+c+d=1 \right\}$$

3. $$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ in } \mathbb{R}^4 \,\middle|\, abcd \neq 0 \right\}$$

4. $$\left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \text{ in } \mathbb{R}^4 \,\middle|\, a, b, c, d \text{ rational} \right\}$$

Subspaces are spans

Fact. Every subspace V is equal to some span.

We already said that all spans were subspaces, so now we know that three things are the same:

- subspaces
- spans
- planes through 0

Column Space and Null Space

$A = m \times n$ matrix.

$\mathrm{Col}(A)$ = column space of A = span of the columns of A = range of T_A

$\mathrm{Nul}(A)$ = null space of A = set of solutions to $Ax = 0$

Example. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

Column Space and Null Space

$A = m \times n$ matrix.

$\text{Col}(A) =$ subspace of $\mathbb{R}^m$

$\text{Nul}(A) =$ subspace of $\mathbb{R}^n$

Note that it is easier to check that $\text{Nul}(A)$ is a subspace than it is to check that $\text{Nul}(A)$ is a span.

Bases

$V =$ subspace of $\mathbb{R}^n$

A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V =$ span$\{v_1, \ldots, v_k\}$
2. the v_i are linearly independent

$\dim(V) =$ dimension of $V = k$

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$?

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Fact. If $A = n \times n$ matrix, then:

$$A \text{ is invertible} \Leftrightarrow \mathrm{Col}(A) = \mathbb{R}^n$$

Bases for planes

Q. Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Section 2.8 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Subspaces are the same as spans are the same as planes through 0
- Two important subspaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{span}\{v_1, \ldots, v_k\}$
  2. the v_i are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)