Announcements: Oct 16

- Midterm 2 on Friday: 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, & 2.9
- WebWork 2.8 & 2.9 due Wednesday
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Review Sessions TBA

Section 2.8

Subspaces of $\mathbb{R}^n$

Subspaces

A subspace of $\mathbb{R}^n$ is a subset V with:

1. The zero vector is in V.
2. If u and v are in V, then $u + v$ is also in V.
3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.

These three things are the same:

- subspaces
- spans
- planes through 0

Column Space and Null Space

$A = m \times n$ matrix.

$$\begin{aligned}\mathrm{Col}(A) &= \text{column space of } A = \text{span of the columns of } A = \text{range of } T_A \\ &= \text{subspace of } \mathbb{R}^m\end{aligned}$$

$$\begin{aligned}\mathrm{Nul}(A) &= \text{null space of } A = \text{set of solutions to } Ax = 0 \\ &= \text{subspace of } \mathbb{R}^n\end{aligned}$$

Example. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

Bases

V = subspace of $\mathbb{R}^n$

A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that

1. $V = \text{span}\{v_1, \ldots, v_k\}$
2. the v_i are linearly independent

$\dim(V)$ = dimension of $V = k$

Q. What is one basis for $\mathbb{R}^2$? $\mathbb{R}^n$?

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Bases for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a basis for $\mathrm{Nul}(A)$
- the pivot columns of A form a basis for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Fact. If $A = n \times n$ matrix, then:

$$A \text{ is invertible} \Leftrightarrow \mathrm{Col}(A) = \mathbb{R}^n$$

Bases for planes

Q. Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Section 2.8 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Subspaces are the same as spans are the same as planes through 0
- Two important subspaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- A basis for V is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{span}\{v_1, \ldots, v_k\}$
  2. the v_i are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)

Section 2.9

Dimension and Rank

Bases as Coordinate Systems

V = subspace of $\mathbb{R}^n$

$B = \{b_1, b_2, \dots, b_k\}$ is a basis for V

x a vector in V

Then we can write x uniquely as

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k$$

We write

$$[x]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

These are the B-coordinates of x.

Bases as Coordinate Systems

Example

Say $b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$B = \{b_1, b_2\}$

$V = \text{Span}\{b_1, b_2\}$.

Q. Verify that B is a basis for V and find the B-coordinates of $x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix}$

Bases as Coordinate Systems

Example

Say $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$

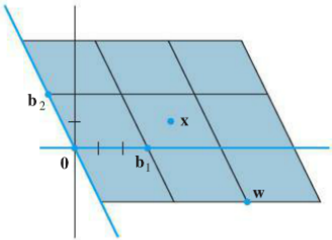
$V = \text{Span}\{v_1, v_2, v_3\}$.

Q. Find a basis for V and find the B-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$

Bases as Coordinate Systems

Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$



Use the figure to estimate the B-coordinates of

$$w = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \text{ and } x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

Rank Theorem

$$\begin{aligned} \text{rank}(A) &= \dim \text{Col}(A) = \#\ \text{pivot columns} \\ \dim \text{Nul}(A) &= \#\ \text{non} - \text{pivot columns} \end{aligned}$$

Rank-Nullity Theorem. $\text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A)$

Example. $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

Poll

Poll

If A and B are 3×3 matrices, and $\operatorname{rank}(A) = \operatorname{rank}(B) = 2$ then what are the possible values of $\operatorname{rank}(AB)$?

a) 0

b) 1

c) 2

d) 3

e) 4

Two More Theorems

Basis Theorem

If V is a k-dimensional subspace of $\mathbb{R}^n$, then

- any k linearly independent vectors of V form a basis for V
- any k vectors that span V form a basis for V

In other words if a set has two of these three properties, it is a basis:

spans V, linearly independent, k vectors

Two More Theorems

Invertible Matrix Theorem

(a) A is invertible

$\vdots$

(m) cols of A form a basis for $\mathbb{R}^n$

(n) $\mathrm{Col}(A) = \mathbb{R}^n$

(o) $\dim \mathrm{Col}(A) = n$

(p) $\mathrm{rank}(A) = n$

(q) $\mathrm{Nul}(A) = \{0\}$

(r) $\dim \mathrm{Nul}(A) = 0$

Sections 2.8/9 Summary

- A subspace of $\mathbb{R}^n$ is a non-empty subset closed under linear combinations.
- Two important subspaces are
  - $\mathrm{Col}(A)$ = span of columns of A.
  - $\mathrm{Nul}(A)$ = (solutions to $Ax = 0$).
- A basis for a subspace W is a set of lin. ind. vectors that spans W.
  - To find the B–coords of u, solve $Bx = u$
- The dimension of a subspace is the number of elements in the basis.
- Use row reduction to find a basis for Col(A) or Nul(A).
  - Pivot columns of A give a basis for Col(A).
  - Parametric form gives a basis for Nul(A).

Rank-Nullity Theorem. $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \#\mathrm{cols}(A)$

Basis Theorem. Suppose V is a k-dimensional subspace of $\mathbb{R}^n$. Then

- Any k linearly independent vectors in V form a basis for V.
- Any k vectors in V that span V form a basis.