Announcements: Oct 16

- Midterm 2 on Friday: 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, & 2.9
- WebWork 2.8 & 2.9 due Wednesday
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Review Sessions TBA
Section 2.8

Subspaces of $\mathbb{R}^n$
Subspaces

A subspace of $\mathbb{R}^n$ is a subset $V$ with:

1. The zero vector is in $V$.
2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.

These three things are the same:

- subspaces
- spans
- planes through 0
Column Space and Null Space

\[ A = m \times n \text{ matrix.} \]

\[ \text{Col}(A) = \text{column space of } A = \text{span of the columns of } A = \text{range of } T_A \]
\[ = \text{subspace of } \mathbb{R}^m \]

\[ \text{Nul}(A) = \text{null space of } A = \text{set of solutions to } Ax = 0 \]
\[ = \text{subspace of } \mathbb{R}^n \]

Example. \[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \]
Bases

\[ V = \text{subspace of } \mathbb{R}^n \]

A basis for \( V \) is a set of vectors \( \{v_1, v_2, \ldots, v_k\} \) such that

1. \( V = \text{span}\{v_1, \ldots, v_k\} \)
2. the \( v_i \) are linearly independent

\[ \dim(V) = \text{dimension of } V = k \]

Q. What is one basis for \( \mathbb{R}^2? \) \( \mathbb{R}^n? \)
Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

Find bases for $\text{Nul}(A)$ and $\text{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$
Bases for $\text{Nul}(A)$ and $\text{Col}(A)$

In general:
- our usual parametric solution for $Ax = 0$ gives a basis for $\text{Nul}(A)$
- the pivot columns of $A$ form a basis for $\text{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Fact. If $A = n \times n$ matrix, then:

\[ A \text{ is invertible} \iff \text{Col}(A) = \mathbb{R}^n \]
Bases for planes

Q. Find a basis for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$. 
Section 2.8 Summary

- A **subspace** of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.

- Subspaces are the same as spans are the same as planes through 0

- Two important subspaces $\text{Nul}(A)$ and $\text{Col}(A)$

- A **basis** for $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{span}\{v_1, \ldots, v_k\}$
  2. the $v_i$ are linearly independent

- The number of vectors in a basis for a subspace is the dimension.

- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form

- Find a basis for $\text{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)
Section 2.9
Dimension and Rank
Bases as Coordinate Systems

\( V = \text{subspace of } \mathbb{R}^n \)

\( B = \{b_1, b_2, \ldots, b_k\} \) is a basis for \( V \)

\( x \) a vector in \( V \)

Then we can write \( x \) uniquely as

\[
x = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k
\]

We write

\[
[x]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}
\]

These are the B-coordinates of \( x \).
Bases as Coordinate Systems

Example

Say \( b_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \), \( b_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)

\( B = \{b_1, b_2\} \)

\( V = \text{Span}\{b_1, b_2\} \).

Q. Verify that \( B \) is a basis for \( V \) and find the \( B \)-coordinates of \( x = \begin{pmatrix} 5 \\ 3 \\ 5 \end{pmatrix} \).
Bases as Coordinate Systems

Example

Say \( \mathbf{v}_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} \)

\( V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \).

Q. Find a basis for \( V \) and find the \( B \)-coordinates of \( \mathbf{x} = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix} \).
Bases as Coordinate Systems

Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

Use the figure to estimate the $B$-coordinates of $w = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$ and $x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.
**Rank Theorem**

\[
\text{rank}(A) = \text{dim Col}(A) = \# \text{ pivot columns} \\
\text{dim Nul}(A) = \# \text{ non - pivot columns}
\]

**Rank-Nullity Theorem.** \( \text{rank}(A) + \text{dim Nul}(A) = \#\text{cols}(A) \)

**Example.**  
\[ A = \begin{pmatrix}  1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \]
If $A$ and $B$ are $3 \times 3$ matrices, and $\text{rank}(A) = \text{rank}(B) = 2$ then what are the possible values of $\text{rank}(AB)$?

a) 0
b) 1
c) 2
d) 3
e) 4
Two More Theorems

Basis Theorem
If $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$, then

- any $k$ linearly independent vectors of $V$ form a basis for $V$
- any $k$ vectors that span $V$ form a basis for $V$

In other words if a set has two of these three properties, it is a basis:

spans $V$, linearly independent, $k$ vectors
Two More Theorems

Invertible Matrix Theorem

(a) $A$ is invertible

: 

(m) cols of $A$ form a basis for $\mathbb{R}^n$

(n) $\text{Col}(A) = \mathbb{R}^n$

(o) $\dim \text{Col}(A) = n$

(p) $\text{rank}(A) = n$

(q) $\text{Nul}(A) = \{0\}$

(r) $\dim \text{Nul}(A) = 0$
Sections 2.8/9 Summary

- **A subspace** of $\mathbb{R}^n$ is a non-empty subset closed under linear combinations.
- Two important subspaces are
  - $\text{Col}(A) = \text{span of columns of } A$.
  - $\text{Nul}(A) = (\text{solutions to } Ax = 0)$.
- **A basis** for a subspace $W$ is a set of lin. ind. vectors that spans $W$.
  - To find the $B$–coords of $u$, solve $Bx = u$
- The **dimension** of a subspace is the number of elements in the basis.
- Use row reduction to find a basis for $\text{Col}(A)$ or $\text{Nul}(A)$.
  - Pivot columns of $A$ give a basis for $\text{Col}(A)$.
  - Parametric form gives a basis for $\text{Nul}(A)$.

**Rank-Nullity Theorem.** $\text{rank}(A) + \text{dim Nul}(A) = \#\text{cols}(A)$

**Basis Theorem.** Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then
- Any $k$ linearly independent vectors in $V$ form a basis for $V$.
- Any $k$ vectors in $V$ that span $V$ form a basis.