Announcements: Oct 18

- Midterm 2 on Friday: 1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, & 2.9
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Review Sessions
  - Martin, Thu 5:30, Skiles 254
  - Arjun, Thu 8:30-9:30, Skiles 249
Section 2.8 Summary

- A subspace of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.

- Subspaces are the same as spans are the same as planes through 0

- Two important subspaces $\text{Nul}(A)$ and $\text{Col}(A)$

- A basis for $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{span}\{v_1, \ldots, v_k\}$
  2. the $v_i$ are linearly independent

- The number of vectors in a basis for a subspace is the dimension.

- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form

- Find a basis for $\text{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)
Section 2.9
Dimension and Rank
Bases as Coordinate Systems

\[ V = \text{subspace of } \mathbb{R}^n \]

\( B = \{b_1, b_2, \ldots, b_k\} \) is a basis for \( V \)

\( x \) a vector in \( V \)

Then we can write \( x \) uniquely as

\[ x = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k \]

We write

\[
[x]_B = \begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_k
\end{pmatrix}
\]

These are the B-coordinates of \( x \).
Bases as Coordinate Systems

Example

Say \( v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}, \ v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}, \ v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix} \)

\( V = \text{Span}\{v_1, v_2, v_3\}. \)

Q. Find a basis for \( V \) and find the \( B \)-coordinates of \( x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix} \).
Bases as Coordinate Systems

Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

Use the figure to estimate the $B$-coordinates of $w = \begin{pmatrix} 7 \\ -2 \end{pmatrix}$ and $x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.
Rank Theorem

\[
\text{rank}(A) = \dim \text{Col}(A) = \# \text{ pivot columns} \\
\dim \text{Nul}(A) = \# \text{ non-pivot columns}
\]

Rank-Nullity Theorem. \(\text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A)\)

Example. \(A = \begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}\)
Two More Theorems

Basis Theorem
If $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$, then

- any $k$ linearly independent vectors of $V$ form a basis for $V$
- any $k$ vectors that span $V$ form a basis for $V$

In other words if a set has two of these three properties, it is a basis:

spans $V$, linearly independent, $k$ vectors
Two More Theorems

Invertible Matrix Theorem

(a) $A$ is invertible

(m) cols of $A$ form a basis for $\mathbb{R}^n$

(n) $\text{Col}(A) = \mathbb{R}^n$

(o) $\text{dim} \text{Col}(A) = n$

(p) $\text{rank}(A) = n$

(q) $\text{Nul}(A) = \{0\}$

(r) $\text{dim} \text{Nul}(A) = 0$
Sections 2.8/9 Summary

• A **subspace** of $\mathbb{R}^n$ is a non-empty subset closed under linear combinations.
  
  • Two important subspaces are
    
    ▶ $\text{Col}(A) = \text{span}$ of columns of $A$.
    ▶ $\text{Nul}(A) = (\text{solutions to } Ax = 0)$.
  
  • A **basis** for a subspace $W$ is a set of lin. ind. vectors that spans $W$.
    
    ▶ To find the $B$–coords of $u$, solve $Bx = u$

  • The **dimension** of a subspace is the number of elements in the basis.
  
  • Use row reduction to find a basis for $\text{Col}(A)$ or $\text{Nul}(A)$.
    
    ▶ Pivot columns of $A$ give a basis for $\text{Col}(A)$.
    ▶ Parametric form gives a basis for $\text{Nul}(A)$.

**Rank-Nullity Theorem.** $\text{rank}(A) + \text{dim Nul}(A) = \#\text{cols}(A)$

**Basis Theorem.** Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then

• Any $k$ linearly independent vectors in $V$ form a basis for $V$.
• Any $k$ vectors in $V$ that span $V$ form a basis.
Invertibility and linear transformations

Which of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are invertible?

1. orthogonal projection to the $xy$-plane
   $T(x, y, z) = (x, y, 0)$

2. the zero transformation $T(0) = 0$

3. reflection about the $yz$-plane
   $T(x, y, z) = (-x, y, z)$

4. $T(x, y, z) = (z, z, z)$

5. rotation about the $z$-axis by $\pi/2$,
   $T(x, y, z) = (y, -x, z)$

Extra: Which are one-to-one, onto? Find the matrices.
Suppose that $A$ and $B$ have the same column space and the same null space. Must it be true that $A = B$? Explain your answer.
Important terms

- linearly independent
- linear transformation
- one-to-one
- onto
- invertibility (for a matrix and a linear transformation)
- subspace
- basis
- dimension
- column space
- null space
Some Questions

Suppose $A$ is a square matrix and $U(v) = A^T v$ is onto. What is $\text{Nul } A$?

Suppose $A$ is a $4 \times 3$ matrix and the range of $T(v) = Av$ is a line. What is the dimension of $\text{Nul } A$?

Suppose $A$ is a $4 \times 5$ matrix. Can it be that $\text{Col } A$ is $\mathbb{R}^4$?

Suppose $A$ is a $5 \times 6$ matrix. Can $T(v) = Av$ be one-to-one? onto?

Find an example of a matrix whose null space is the span of $(1, 1, 1)$ in $\mathbb{R}^3$. 