# Announcements: Oct 18

- Midterm 2 on Friday: 1.7, 1.8, 1.9, 2.1, 2.2, 2.3, 2.8, & 2.9
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230
- Review Sessions
  - Martin, Thu 5:30, Skiles 254
  - Arjun, Thu 8:30-9:30, Skiles 249

## Section 2.8 Summary

- A subspace of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.
- Subspaces are the same as spans are the same as planes through 0
- Two important subspaces $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- A basis for $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{span}\{v_1, \ldots, v_k\}$
  2. the $v_i$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\mathrm{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)

# Section 2.9

## Dimension and Rank

# Bases as Coordinate Systems

$V$ = subspace of $\mathbb{R}^n$

$B = \{b_1, b_2, \ldots, b_k\}$ is a basis for $V$

$x$ a vector in $V$

Then we can write $x$ uniquely as

$$x = c_1 b_1 + c_2 b_2 + \cdots + c_k b_k$$

We write

$$[x]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{pmatrix}$$

These are the B-coordinates of $x$.

# Bases as Coordinate Systems

## Example

Say $v_1 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$, $v_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$, $v_3 = \begin{pmatrix} 2 \\ 8 \\ 6 \end{pmatrix}$

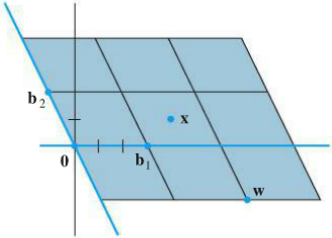
$V = \mathrm{Span}\{v_1, v_2, v_3\}$.

*Q*. Find a basis for $V$ and find the $B$-coordinates of $x = \begin{pmatrix} 4 \\ 11 \\ 8 \end{pmatrix}$

# Bases as Coordinate Systems

Consider the following basis for $\mathbb{R}^2$:

$$B = \left\{ \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$




Use the figure to estimate the $B$-coordinates of

$$w = \begin{pmatrix} 7 \\ -2 \end{pmatrix} \text{ and } x = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

# Rank Theorem

$$\begin{aligned} \text{rank}(A) &= \dim \text{Col}(A) = \#\ \text{pivot columns} \\ \dim \text{Nul}(A) &= \#\ \text{non} - \text{pivot columns} \end{aligned}$$

Rank-Nullity Theorem. $\text{rank}(A) + \dim \text{Nul}(A) = \#\text{cols}(A)$

*Example.* $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

# Two More Theorems

## Basis Theorem

If $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$, then

- any $k$ linearly independent vectors of $V$ form a basis for $V$
- any $k$ vectors that span $V$ form a basis for $V$

In other words if a set has two of these three properties, it is a basis:

spans $V$, linearly independent, $k$ vectors

# Two More Theorems

## Invertible Matrix Theorem

(a) $A$ is invertible

$\vdots$

(m) cols of $A$ form a basis for $\mathbb{R}^n$

(n) $\mathrm{Col}(A) = \mathbb{R}^n$

(o) $\dim \mathrm{Col}(A) = n$

(p) $\mathrm{rank}(A) = n$

(q) $\mathrm{Nul}(A) = \{0\}$

(r) $\dim \mathrm{Nul}(A) = 0$

# Sections 2.8/9 Summary

- A subspace of $\mathbb{R}^n$ is a non-empty subset closed under linear combinations.
- Two important subspaces are
  - $\mathrm{Col}(A)$ = span of columns of $A$.
  - $\mathrm{Nul}(A)$ = (solutions to $Ax = 0$).
- A basis for a subspace $W$ is a set of lin. ind. vectors that spans $W$.
  - To find the $B$–coords of $u$, solve $Bx = u$
- The dimension of a subspace is the number of elements in the basis.
- Use row reduction to find a basis for Col($A$) or Nul($A$).
  - Pivot columns of $A$ give a basis for Col($A$).
  - Parametric form gives a basis for Nul($A$).

Rank-Nullity Theorem. $\mathrm{rank}(A) + \dim \mathrm{Nul}(A) = \#\mathrm{cols}(A)$

Basis Theorem. Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then

- Any $k$ linearly independent vectors in $V$ form a basis for $V$.
- Any $k$ vectors in $V$ that span $V$ form a basis.

# Invertibility and linear transformations

**Poll**

Which of the following linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^3$ are invertible?

1. orthogonal projection to the $xy$-plane
   $T(x, y, z) = (x, y, 0)$
2. the zero transformation $T(0) = 0$
3. reflection about the $yz$-plane
   $T(x, y, z) = (-x, y, z)$
4. $T(x, y, z) = (z, z, z)$
5. rotation about the $z$-axis by $\pi/2$,
   $T(x, y, z) = (y, -x, z)$

Extra: Which are one-to-one, onto? Find the matrices.

## Column space and Null space

Suppose that $A$ and $B$ have the same column space and the same null space. Must it be true that $A = B$? Explain your answer.

## Important terms

- linearly independent
- linear transformation
- one-to-one
- onto
- invertibility (for a matrix and a linear transformation)
- subspace
- basis
- dimension
- column space
- null space

## Some Questions

Suppose $A$ is a square matrix and $U(v) = A^T v$ is onto. What is $\operatorname{Nul} A$?

Suppose $A$ is a $4 \times 3$ matrix and the range of $T(v) = Av$ is a line. What is the dimension of $\operatorname{Nul} A$?

Suppose $A$ is a $4 \times 5$ matrix. Can it be that $\operatorname{Col} A$ is $\mathbb{R}^4$?

Suppose $A$ is a $5 \times 6$ matrix. Can $T(v) = Av$ be one-to-one? onto?

Find an example of a matrix whose null space is the span of $(1, 1, 1)$ in $\mathbb{R}^3$.