Announcements: Nov 1

- WebWork 3.1, 3.2 due today
- Quiz 3.1, 3.2 Friday
- Midterm 3 on Friday Nov 17
- Upcoming Office Hours
  - Me: Wednesday 2-3, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other sources of help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 5, 2-4 Clough 102
    - Sun Nov 12, 2-4, Clough 102
    - Sun Nov 19, 2-4, Clough 102

Orientation

Last time:

- Definition of eigenvalue/eigenvector
- How to confirm an eigenvector
- How to find an eigenspace for a given eigenvalue

Today:

- How to find the eigenvalues, via the characteristic polynomial
- Algebraic multiplicity
- Similar matrices

Eigenvectors and Eigenvalues

If A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Eigenvalues

Poll

True or False? Every square matrix has an eigenvalue.

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Suppose A corresponds to reflection about the line $y = -x$. What are the eigenvalues/eigenvectors of A?

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Suppose A corresponds to rotation of $\mathbb{R}^2$ by $\pi/2$. What are the eigenvalues/eigenvectors of A?

Section 5.2

The characteristic polynomial

5.2 The characteristic polynomial

Outline

- The characteristic polynomial: a systematic way to find eigenvalues
  - 2×2 matrices
  - 3×3 matrices
- algebraic multiplicity of eigenvalues
- similar matrices $\rightsquigarrow$ same eigenvalues

Characteristic polynomial

Recall:

$$\lambda \text{ is an eigenvalue of } A \Leftrightarrow A - \lambda I \text{ is not invertible}$$

So to find eigenvalues of A we solve

$$\det(A - \lambda I) = 0$$

The left hand side is a polynomial called the characteristic polynomial of A.

The roots of the characteristic polynomial are the eigenvalues of A.

Characteristic polynomial

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} 5 & 2 \\ 2 & 1 \end{pmatrix}$$

Characteristic polynomial

2×2 matrices

Find the characteristic polynomial and eigenvalues of

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Characteristic polynomials

3×3 matrices

Find the characteristic polynomial of the rabbit population matrix.

$$\begin{pmatrix} 0 & 6 & 8 \\ \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

What are the eigenvalues?

Hint: We already know one eigenvalue!

Eigenvalues

Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Algebraic multiplicity

The algebraic multiplicity of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example. Find the algebraic multiplicities of the eigenvalues for

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Similar matrices

Two $n \times n$ matrices A and B are similar if there is a matrix C so that

$$A = CBC^{-1}$$

A is doing in the C-basis what B does in the regular basis.

Similar matrices

And the characteristic polynomial

Fact. If A and B similar, they have the same characteristic polynomial.

Similar matrices

Example

Similar: $A = CBC^{-1}$

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

A is doing in the C-basis what B does in the regular basis.

Demo

Non-Eigenvectors

What does A do to non-eigenvectors?

Example. $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$.

Almost all vectors get pulled to the x-axis, which is the eigenvector with largest eigenvalue.

Eigenvalues

Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?

Eigenvalues

Summary

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- The number λ is an eigenvalue of $A \leftrightarrow \det(A - \lambda I) = 0$
- The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Similar matrices have the same characteristic polynomial and eigenvalues
- Eigenvectors with different eigenvalues are linearly independent