Announcements: Nov 6

- WebWork 5.1, 5.2 due Wednesday
- Quiz 5.1, 5.2 Friday
- Midterm 3 on Friday Nov 17
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 2-3, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 12, 2-4, Clough 102
    - Sun Nov 19, 2-4, Clough 102

Orientation

Last week:

- Eigenvectors & eigenvalues
- (Similar matrices)

Today:

- Diagonalization
- Using diagonalization to take powers
- Algebraic versus geometric dimension

Eigenvectors and Eigenvalues

If A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A

Similar matrices

Two $n \times n$ matrices A and B are similar if there is a matrix C so that

$$A = CBC^{-1}$$

A is doing in the C-basis what B does in the regular basis.

Fact. If A and B similar, they have the same characteristic polynomial.

Similar matrices

Example

Similar: $A = CBC^{-1}$

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

A is doing in the C-basis what B does in the regular basis.

Demo

Section 5.3

Diagonalization

5.3 Diagonalization

Outline

- Taking powers of diagonal matrices is easy
- Taking powers of diagonalizable matrices is still easy
- Algebraic multiplicity vs geometric multiplicity vs diagonalizability
- Application: networks

Powers of diagonal matrices

We have see that it is useful to take powers of matrices: for instance in computing rabbit populations.

If A is diagonal, A^k is easy to compute. For example:

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}^{10}$$

Powers of matrices that are similar to diagonal ones

What if A is not diagonal? Suppose we need to compute

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}^{10}$$

What would we do?

Earlier in the notes, we saw this matrix is similar to a diagonal one:

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$$

"diagonalization"

So...

Diagonalization

Suppose A is $n \times n$. We say that A is diagonalizable if it is similar to a diagonal matrix:

$$A = CDC^{-1} \qquad\qquad D = \text{diagonal}$$

How does this factorization of A help describe what A does to $\mathbb{R}^n$?

Diagonalization

Theorem. A is diagonalizable $\Leftrightarrow A$ has n linearly independent eigenvectors.

In this case

$$A = (v_1 \ v_2 \cdots v_n) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} (v_1 \ v_2 \cdots v_n)^{-1}$$

where $v_1, \ldots, v_n$ are linearly independent eigenvectors and $\lambda_1, \ldots, \lambda_n$ are the corresponding eigenvalues (in order).

Why?

Example

Diagonalize if possible.

$$\begin{pmatrix} 2 & 6 \\ 0 & -1 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Example

Diagonalize if possible.

$$\begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{pmatrix}$$

▸ Demo

More Examples

Diagonalize if possible.

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

Poll

Poll

Which are diagonalizable?

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

Distinct Eigenvalues

Fact. If A has n distinct eigenvalues, then A is diagonalizable.

Why?

Non-Distinct Eigenvalues

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

Application: Business

Say your car rental company has 3 locations. Make a matrix M whose ij entry is the probability that a car at location i ends at location j. For example,

$$M = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Note the columns sum to 1. The eigenvector with eigenvalue 1 is the steady-state. Any other vector gets pulled to this state. Applying powers of M gives the state after some number of iterations.

Why is this similar/same as Google?

Summary

- A is diagonalizable if it is similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable