Announcements: Nov 8

- Need another MatLab volunteer!
- WebWork 5.1, 5.2 due Wednesday
- Quiz 5.1, 5.2 Friday
- Midterm 3 on Friday Nov 17
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 12, 2-4, Clough 102
    - Sun Nov 19, 2-4, Clough 102

Algebraic versus Geometric Multiplicity

Theorem. Suppose

- $A = n \times n$, has eigenvalues $\lambda_1, \ldots, \lambda_k$
- a_i = algebraic multiplicity of λ_i
- d_i = dimension of λ_i eigenspace ("geometric multiplicity")

Then

1. $d_i \leq a_i$ for all i
2. A is diagonalizable $\Leftrightarrow \Sigma d_i = n$
   $\Leftrightarrow \Sigma a_i = n$ and $d_i = a_i$ for all i

Example.

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

Applications: Rental Cars, Redbox, and Google

A car rental company has 3 locations. Make a matrix M whose ij entry is the fraction of cars at location i ending at location j.

$$M = \begin{pmatrix} .3 & .4 & .5 \\ .3 & .4 & .3 \\ .4 & .2 & .2 \end{pmatrix}$$

Poll. Which rental car location needs the most cars?

Why is this similar/same as Google?

▸ Demo

Section 5.5

Complex Eigenvalues

Outline

- Rotation matrices have no eigenvectors
- Crash course in complex numbers
- Finding complex eigenvectors and eigenvalues
- Complex eigenvalues correspond to rotations + dilations

A matrix without an eigenvector

Recall the rotation matrix:

$$A = 1/\sqrt{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

This matrix has no eigenvectors. Why?

Imaginary numbers

Problem. When solving polynomial equations, we often run up against the issue that we can't take the square root of a negative number:

$$x^2 + 1 = 0$$

Solution. Take square roots of negative numbers:

$$x = \pm\sqrt{-1}$$

We usually write $\sqrt{-1}$ as i (for "imaginary"), so $x = \pm i$.

Now try solving these:

$$x^2 + 3 = 0$$

$$x^2 - x + 1 = 0$$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can identify $\mathbb{C}$ with $\mathbb{R}^2$: $\quad a + bi \leftrightarrow (a, b)$

We can add/multiply (and divide!) complex numbers:

$(2 - 3i) + (-1 + i) =$

$(2 - 3i)(-1 + i) =$

Complex numbers

The complex numbers are the numbers

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

We can conjugate complex numbers: $\overline{a + bi} = a - bi$

We can take absolute values of complex numbers:
$|a + bi| = \sqrt{a^2 + b^2}$

We can write complex numbers in polar coordinates:
$r(\cos\theta + i\,\sin\theta)$ (this makes it easier to multiply/divide)

Complex numbers and polynomials

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

Fact. Every quadratic polynomial has two complex roots.

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots.

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

So what are the possibilities for degree 2, 3 polynomials?

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Summary

- Complex numbers allow us to solve all polynomials completely, and find n eigenvectors for an $n \times n$ matrix
- Multiplying by a complex number rotates/scales the complex plane
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$