Announcements: Nov 10

- WebWork 5.3, 5.5 due Wednesday
- Midterm 3 on Friday Nov 17
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 19, 2-4, Clough 102

Orientation

Last time we learned about complex numbers. This allowed us to find complex eigenvalues and eigenvectors for matrices where previously we could not find eigenvalues.

We understand real eigenvalues as describing the amount of scaling along a direction. We will understand complex eigenvalue as scaling/rotation on a two-dimensional plane.

Section 5.5

Complex Eigenvalues

Complex numbers and polynomials

$$\mathbb{C} = \{a + bi \mid a, b \text{ in } \mathbb{R}\}$$

Fundamental theorem of algebra. Every polynomial of degree n has exactly n complex roots (with multiplicity).

We can now find complex eigenvectors and eigenvalues.

Fact. If λ is an eigenvalue of A with eigenvector v then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{v}$.

So what are the possibilities for 2×2 and 3×3 matrices?

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

What relationship is there between $T(v) = Av$ and the eigenvalues of A?

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

Three shortcuts for complex eigenvectors

Suppose we have a 2×2 matrix with complex eigenvalue λ.

(1) We do not need to row reduce $A - \lambda I$ by hand; we know the bottom row will become zero.

(2) Then if the reduced matrix is:

$$A = \begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix}$$

the eigenvector is

$$A = \begin{pmatrix} -y \\ x \end{pmatrix}$$

(3) Also, we get the other eigenvalue/eigenvector pair for free: conjugation.

A 3×3 example

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}$$

What do complex eigenvalues mean?

With n real eigenvectors, we have a picture for what the matrix does to $\mathbb{R}^n$.

What about complex eigenvectors? What does the matrix do to $\mathbb{R}^n$?

We saw that rotation matrices have complex eigenvalues. Do complex eigenvalues always correspond to rotations?

Almost...

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is *similar to* the product of a rotation and a dilation.

What do complex eigenvalues mean?

Fact. If an $n \times n$ matrix A has a complex eigenvalue there is a 2D plane in $\mathbb{R}^n$ where A is *similar to* the product of a rotation and a dilation.

Here is the actual statement for 2×2 matrices:

Theorem. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = CBC^{-1}$$

where

$$C = \begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

B is the composition of a rotation by θ and scaling by r.

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B is the composition of a rotation by θ and scaling by r. Why?

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Example. Find C and B when

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

Example

Compute the decomposition CBC^{-1} for the matrix:

$$A = \begin{pmatrix} 5 & -2 \\ 1 & 3 \end{pmatrix}$$

Three pictures

There are three possible pictures for the action on $\mathbb{R}^2$ of a 2×2 matrix with complex eigenvalues.

$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$	$\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}$	$\begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}$
$\lambda = 1 + i$	$\lambda = 3/5 + 4/5i$	$\lambda = 1/2 + 1/2i$
$\lvert\lambda\rvert > 1$	$\lvert\lambda\rvert = 1$	$\lvert\lambda\rvert < 1$
spiral in	rotate around ellipse	spiral out
Demo	Demo	Demo

What about the higher dimensional case?

Theorem. Let A be a real 3×3 matrix. Suppose that it has one real eigenvalue α and two complex eigenvalues λ and $\overline{\lambda}$. We can find a "block diagonalization"

$$A = CBC^{-1}$$

where:

$B =$

$C =$

The $n \times n$ case is similar.

A 3×3 example

Find the block diagonalization of:

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

A 3×3 example

What does A do to $\mathbb{R}^3$? Draw a picture!

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

Demo

Summary of complex eigenvalues

- An $n \times n$ matrix has n complex eigenvalues (with multiplicity).
- Complex eigenvalues/eigenvectors come in conjugate pairs.
- If a matrix has a complex eigenvalue, then there is a plane in $\mathbb{R}^n$ on which A is similar to a scale/rotation.
- Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

  $$A = CBC^{-1}$$

  where

  $$C = \begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

  So the amount of rotation/scaling is determined by λ.
- The 3×3 case is similar.