Announcements: Nov 15

- Midterm 3 on Friday
- Upcoming Office Hours
  - Me: Wednesday 1-2, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 19, 2-4, Clough 102

Review Sessions
  - Martin Thursday 7-9 Skiles 257
  - Qianli Thursday 1-2 Skiles 271
A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

\[ A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix} \]
What do complex eigenvalues mean?

Theorem. Let $A$ be a real $2 \times 2$ matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector $v$. Then

$$A = CBC^{-1}$$

where

$$C = (\text{Re} \ v \quad \text{Im} \ v) \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Example. Find $C$ and $B$ when

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

By how much does $B$ rotate/scale?
A $3 \times 3$ example

Find the block diagonalization of:

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

What does $A$ do to $\mathbb{R}^3$? Draw a picture!
Summary of complex eigenvalues

- An $n \times n$ matrix has $n$ complex eigenvalues (with multiplicity).
- Complex eigenvalues/eigenvectors come in conjugate pairs.
- If a matrix has a complex eigenvalue, then there is a plane in $\mathbb{R}^n$ on which $A$ is similar to a scale/rotation.
- Let $A$ be a real $2 \times 2$ matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector $v$. Then

$$A = CB C^{-1}$$

where

$$C = (\text{Re} \, v \quad \text{Im} \, v) \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

So the amount of rotation/scaling is determined by $\lambda$.
- The $3 \times 3$ case is similar.
Chapter 6
Orthogonality and Least Squares
Section 6.1

Inner Product, Length, and Orthogonality
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.
Outline

- Dot products
- Dot products and orthogonality
- Orthogonal projection
- A formula for projection onto a line
- Orthogonal complements
Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$u \cdot v = \sum_{i=1}^{n} u_i v_i$$

$$= u_1 v_1 + \cdots + u_n v_n$$

$$= u^T v$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$. 
Dot product

Some properties of the dot product

- \( u \cdot v = v \cdot u \)
- \( (u + v) \cdot w = u \cdot w + v \cdot w \)
- \( (cu) \cdot v = c(u \cdot v) \)
- \( u \cdot u \geq 0 \)
- \( u \cdot u = 0 \iff u = 0 \)
Dot product
and Length

Let $v$ be a vector in $\mathbb{R}^n$

$$\|v\| = \sqrt{v \cdot v}$$

= length (or norm) of $v$

Why? Pythagorean Theorem

Fact. $\|cv\| = c\|v\|$

$v$ is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$. 
Orthogonality

Fact. \( u \perp v \iff u \cdot v = 0 \)

Why? Pythagorean theorem again!

\[
\begin{align*}
u \perp v & \iff \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\
& \iff u \cdot u + v \cdot v = u \cdot u - 2u \cdot v + v \cdot v \\
& \iff u \cdot v = 0
\end{align*}
\]

Problem. Find a vector in \( \mathbb{R}^3 \) orthogonal to \((1, 2, 3)\).
Orthogonal complements

\[ W = \text{subspace of } \mathbb{R}^n \]
\[ W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \} \]

**Question.** What is the orthogonal complement of a line in \( \mathbb{R}^3 \)?

**Facts.**

1. \( W^\perp \) is a subspace of \( \mathbb{R}^n \)
2. \( (W^\perp)^\perp = W \)
3. \( \text{dim } W + \text{dim } W^\perp = n \)
4. If \( W = \text{Span}\{w_1, \ldots, w_k\} \) then
   \[ W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w_i \text{ for all } i \} \]
5. The intersection of \( W \) and \( W^\perp \) is \( \{0\} \).
Orthogonal complements
Finding them

**Problem.** Let $W = \text{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^\perp$.

**Problem.** Let $W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find the equation of the line $W^\perp$. 
Orthogonal complements
Finding them

Problem. Let \( W = \text{Span}\{(1, 1, -1), (-1, 2, 1)\} \). Find the equation of the line \( W^\perp \).

Theorem. \( A = m \times n \) matrix

\[
(\text{Row} A)^\perp = \text{Nul} \ A
\]

Why? \( Ax = 0 \iff x \text{ is orthogonal to each row of } A \)
Orthogonal decomposition

Fact. Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $v$ in $\mathbb{R}^n$ can be written uniquely as

$$v = w + w'$$

where $w \in W$ and $w' \in W^\perp$.

Why? Say that $w_1 + w'_1 = w_2 + w'_2$ where $w_1$ and $w_2$ are in $W$ and $w'_1$ and $w'_2$ are in $W^\perp$. Then $w_1 - w_2 = w'_2 - w'_1$. But the former is in $W$ and the latter is in $W^\perp$, so they must both be equal to 0.

Next time: Find $w$ and $w'$.
Summary

- Basic properties of dot product:
  - $u \cdot u = \|u\|^2$
  - $u \cdot v = 0 \iff u \perp v$

- Orthogonal complements:
  - $W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \}$
  - $(\text{Row } A)^\perp = \text{Nul } A$ (this is how you find $W^\perp$)
  - Every vector $v$ in $\mathbb{R}^n$ can be written uniquely as $v = w + w'$ with $w$ in $W$ and $w' \in W^\perp$