Announcements: Nov 15

- Midterm 3 on Friday
- Upcoming Office Hours
  - Me: Wednesday 1-2, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - Group Tutoring
    - Sun Nov 19, 2-4, Clough 102

  Review Sessions
  - Martin Thursday 7-9 Skiles 257
  - Qianli Thursday 1-2 Skiles 271

Section 5.5

Complex Eigenvalues

A matrix with an eigenvector

Find the eigenvectors and eigenvalues of:

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

What do complex eigenvalues mean?

Theorem. Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

$$A = CBC^{-1}$$

where

$$C = \begin{pmatrix} \mathrm{Re}\, v & \mathrm{Im}\, v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

Example. Find C and B when

$$A = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

By how much does B rotate/scale?

A 3×3 example

Find the block diagonalization of:

$$A = \begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

What does A do to $\mathbb{R}^3$? Draw a picture!

▸ Demo

Summary of complex eigenvalues

- An $n \times n$ matrix has n complex eigenvalues (with multiplicity).
- Complex eigenvalues/eigenvectors come in conjugate pairs.
- If a matrix has a complex eigenvalue, then there is a plane in $\mathbb{R}^n$ on which A is similar to a scale/rotation.
- Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ (where $b \neq 0$) and associated eigenvector v. Then

  $$A = CBC^{-1}$$

  where

  $$C = \begin{pmatrix} \operatorname{Re} v & \operatorname{Im} v \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

  So the amount of rotation/scaling is determined by λ.
- The 3×3 case is similar.

Chapter 6

Orthogonality and Least Squares

Section 6.1

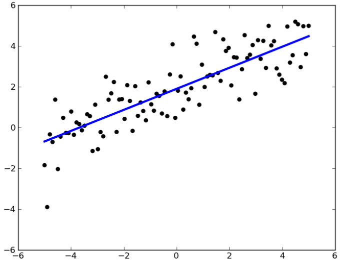
Inner Product, Length, and Orthogonality

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Outline

- Dot products
- Dot products and orthogonality
- Orthogonal projection
- A formula for projection onto a line
- Orthogonal complements

Dot product

Say $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$

$$\begin{aligned} u \cdot v &= \sum_{i=1}^{n} u_i v_i \\ &= u_1 v_1 + \cdots + u_n v_n \\ &= u^T v \end{aligned}$$

Example. Find $(1, 2, 3) \cdot (4, 5, 6)$.

Dot product

Some properties of the dot product

- $u \cdot v = v \cdot u$
- $(u + v) \cdot w = u \cdot w + v \cdot w$
- $(cu) \cdot v = c(u \cdot v)$
- $u \cdot u \geq 0$
- $u \cdot u = 0 \Leftrightarrow u = 0$

Dot product

and Length

Let v be a vector in $\mathbb{R}^n$

$$\begin{aligned} \|v\| &= \sqrt{v \cdot v} \\ &= \text{length (or norm) of } v \end{aligned}$$

Why? Pythagorean Theorem

Fact. $\|cv\| = c\|v\|$

v is a unit vector of $\|v\| = 1$

Problem. Find the unit vector in the direction of $(1, 2, 3, 4)$.

Problem. Find the distance between $(1, 1, 1)$ and $(1, 4, -3)$.

Orthogonality

Fact. $u \perp v \Leftrightarrow u \cdot v = 0$

Why? Pythagorean theorem again!

$$
\begin{aligned}
u \perp v &\Leftrightarrow \|u\|^2 + \|v\|^2 = \|u - v\|^2 \\
&\Leftrightarrow u \cdot u + v \cdot v = u \cdot u - 2u \cdot v + v \cdot v \\
&\Leftrightarrow u \cdot v = 0
\end{aligned}
$$

Problem. Find a vector in $\mathbb{R}^3$ orthogonal to $(1, 2, 3)$.

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Question. What is the orthogonal complement of a line in $\mathbb{R}^3$?

Demo   Demo

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1, 1, -1)\}$. Find the equation of the plane $W^{\perp}$.

Problem. Let $W = \mathrm{Span}\{(1, 1, -1), (-1, 2, 1)\}$. Find the equation of the line $W^{\perp}$.

Orthogonal complements

Finding them

Problem. Let $W = \mathrm{Span}\{(1,1,-1),(-1,2,1)\}$. Find the equation of the line $W^\perp$.

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row} A)^\perp = \mathrm{Nul}\, A$$

Why? $Ax = 0 \Leftrightarrow x$ is orthogonal to each row of A

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector v in $\mathbb{R}^n$ can be written uniquely as

$$v = w + w'$$

where $w \in W$ and $w' \in W^{\perp}$.

Why? Say that $w_1 + w_1' = w_2 + w_2'$ where w_1 and w_2 are in W and w_1' and w_2' are in $W^{\perp}$. Then $w_1 - w_2 = w_2' - w_1'$. But the former is in W and the latter is in $W^{\perp}$, so they must both be equal to 0.

Next time: Find w and w'.

Summary

- Basic properties of dot product:
  - $u \cdot u = \|u\|^2$
  - $u \cdot v = 0 \Leftrightarrow u \perp v$
- Orthogonal complements:
  - $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$
  - $(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$ (this is how you *find* $W^\perp$)
  - Every vector v in $\mathbb{R}^n$ can be written uniquely as $v = w + w'$ with w in W and $w' \in W^\perp$