Announcements: Nov 15

- Come to the front to pick up your Midterm
- Final Exam on Tuesday Dec 12 6:00-8:50pm
- Upcoming Office Hours
  - Me: Monday 1-2
  - Bharat: Tuesday 1:45-2:45, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - CAS Study Session Dec 6 Clough 144/152

  Grades:
  - Midterm 3 average: 74%
  - Overall course average: 85%

Chapter 6

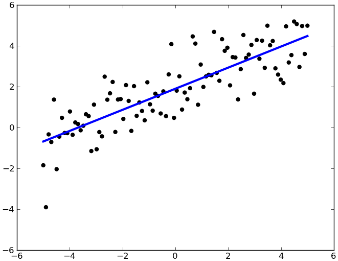
Orthogonality and Least Squares

Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



The answer relies on orthogonality.

Orthogonal complements

W = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\dim W + \dim W^\perp = n$
4. If $W = \mathrm{Span}\{w_1, \ldots, w_k\}$ then
   $W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i\}$
5. The intersection of W and $W^\perp$ is $\{0\}$.

Theorem. $A = m \times n$ matrix

$$(\mathrm{Row}\, A)^\perp = \mathrm{Nul}\, A$$

Orthogonal decomposition

Fact. Say W is a subspace of $\mathbb{R}^n$. Then any vector y in $\mathbb{R}^n$ can be written uniquely as

$$y_W + y_{W^\perp}$$

where $y_W \in W$ and $y_{W^\perp} \in W^\perp$.

▸ Demo

Orthogonal Sets

A set of vectors is orthogonal if each pair of vectors is orthogonal. It is orthonormal if in addition each vector is a unit vector.

Example.

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Fact. An orthogonal set of nonzero vectors is linearly independent.

Why? Suppose $c_i u_1 + c_2 u_2 + c_3 u_3 = 0$. Dot both sides with u_1.

Section 6.2/6.3

Orthogonal Projections

Outline

- Orthogonal projections
- A formula for projecting onto a line
- A formula for projecting onto any subspace
- Projections and best possible solutions

Orthogonal Projections

Let W be a subspace of $\mathbb{R}^n$ and y a vector in $\mathbb{R}^n$.

$$\text{proj}_W(y) = \text{orthogonal projection to } W \text{ of } y$$

Demo

If we write y as $y_W + y_{W^\perp}$ then $\text{proj}_W(y) = y_W$.

Orthogonal projection onto a line

Say $W = \mathrm{Span}\{u\}$.

Fact. $\mathrm{proj}_W(y) = \frac{y \cdot u}{u \cdot u} u$

Demo

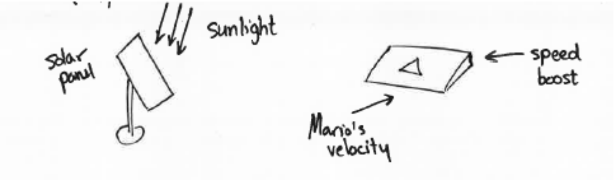
Why? $(y - cu) \cdot u = 0 \Leftrightarrow c = \frac{y \cdot u}{u \cdot u}$

As above, $y - \mathrm{proj}_W(y)$ lies in $W^\perp$.

Problem. Let $u = (1, 2)$ and $W = \langle u \rangle$. Let $y = (1, 1)$.
Write y as $y_W + y_{W^\perp}$.

Orthogonal Projections

Many applications, including:



Orthogonal projection

Projecting onto any subspace

Fact. Say W a subspace of $\mathbb{R}^n$ and y in $\mathbb{R}^n$. We can write y uniquely as:

$$y = y_W + y_{W^\perp}$$

with y_W in W and $y_{W^\perp}$ in $W^\perp$.

Moreover, if $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for W then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

This y_W is $\mathrm{proj}_W(y)$.

Problem. Use the formula to project $(1, 2, 3)$ to the xy-plane.

Orthogonal projection

Projecting onto any subspace

If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for W then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

This y_W is $\mathrm{proj}_W(y)$.

Problem. Let $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ and $y = e_1$. Find y_W.

Orthogonal projection

Matrices for projections

Find A so that $T(v) = Av$ is orthogonal projection onto

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Orthogonal projection

Poll

Suppose $T(v) = Av$ is orthogonal projection onto a plane in $\mathbb{R}^3$. What is A^2 equal to?

1. A
2. A^{-1}
3. $-A$
4. 0
5. I_n
6. $-I_n$

While you are at it: What are the eigenvalues of A?

Orthogonal bases

Finding coordinates with respect to orthogonal bases

Fact. Say that $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = c_1 u_1 + \cdots + c_k u_k$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Same formula as before! Why?

Orthogonal bases

Finding coordinates with respect to orthogonal bases

Fact. Say that $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = c_1 u_1 + \cdots + c_k u_k$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -4 \\ 2 \end{pmatrix} \right\}$$

Orthogonal bases

Finding coordinates with respect to orthogonal bases

Fact. Say that $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace W of $\mathbb{R}^n$ and say that y is in W. Then

$$y = c_1 u_1 + \cdots + c_k u_k$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Problem. Find the B-coordinates of $(6, 1, -8)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Best approximation

W = subspace of $\mathbb{R}^n$

Fact. The projection y_W is the point in W closest to y. In other words:

$$\|y - y_W\| < \|y - w\|$$

for any w in W other than y_W.

Why? Make a right triangle between y, $y - y_W$, and $y_W - w$.

Best approximation

Problem. Find the distance from e_1 to

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

Summary

- $\text{proj}_W(y)$ = orthogonal projection to W of y
- If $\mathcal{B} = \{u_1, \dots, u_k\}$ is an orthogonal basis for W then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

  This y_W is $\text{proj}_W(y)$.
- We find the matrix for projections in the usual way (project the e_i).
- If y is already in W then this gives the $\mathcal{B}$-coordinates.
- The projection of y to W is the closest point in W to y.