Announcements: Nov 15

- Come to the front to pick up your Midterm
- Final Exam on Tuesday Dec 12 6:00-8:50pm
- Upcoming Office Hours
  - Me: Monday 1-2
  - Bharat: Tuesday 1:45-2:45, Skiles 230

Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - CAS Study Session Dec 6 Clough 144/152

Grades:
  - Midterm 3 average: 74%
  - Overall course average: 85%
Chapter 6
Orthogonality and Least Squares
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.
Orthogonal complements

$W = $ subspace of $\mathbb{R}^n$
$W^\perp = \{ v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W \}$

Facts.

1. $W^\perp$ is a subspace of $\mathbb{R}^n$
2. $(W^\perp)^\perp = W$
3. $\text{dim } W + \text{dim } W^\perp = n$
4. If $W = \text{Span}\{w_1, \ldots, w_k\}$ then $W^\perp = \{ v \text{ in } \mathbb{R}^n \mid v \perp w_i \text{ for all } i \}$
5. The intersection of $W$ and $W^\perp$ is $\{0\}$.

Theorem. $A = m \times n$ matrix

$(\text{Row} A)^\perp = \text{Nul } A$
Orthogonal decomposition

**Fact.** Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $y$ in $\mathbb{R}^n$ can be written uniquely as

$$y = y_W + y_{W^\perp}$$

where $y_W \in W$ and $y_{W^\perp} \in W^\perp$. 

▶ Demo
Orthogonal Sets

A set of vectors is **orthogonal** if each pair of vectors is orthogonal. It is **orthonormal** if in addition each vector is a unit vector.

Example.

\[
B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}
\]

Fact. An orthogonal set of nonzero vectors is linearly independent.

Why? Suppose \( c_1u_1 + c_2u_2 + c_3u_3 = 0 \). Dot both sides with \( u_1 \).
Section 6.2/6.3
Orthogonal Projections
Outline

- Orthogonal projections
- A formula for projecting onto a line
- A formula for projecting onto any subspace
- Projections and best possible solutions
Orthogonal Projections

Let $W$ be a subspace of $\mathbb{R}^n$ and $y$ a vector in $\mathbb{R}^n$.

$$\text{proj}_W(y) = \text{orthogonal projection to } W \text{ of } y$$

If we write $y$ as $y_W + y_{W\perp}$ then $\text{proj}_W(y) = y_W$. 
Orthogonal projection onto a line

Say $W = \text{Span}\{u\}$.

**Fact.** $\text{proj}_W(y) = \frac{y \cdot u}{u \cdot u} u$

**Demo**

Why? $(y - cu) \cdot u = 0 \iff c = \frac{y \cdot u}{u \cdot u}$

As above, $y - \text{proj}_W(y)$ lies in $W^\perp$.

**Problem.** Let $u = (1, 2)$ and $W = \langle u \rangle$. Let $y = (1, 1)$. Write $y$ as $y_W + y_{W^\perp}$. 
Orthogonal Projections

Many applications, including:
Orthogonal projection
Projecting onto any subspace

**Fact.** Say $W$ a subspace of $\mathbb{R}^n$ and $y$ in $\mathbb{R}^n$. We can write $y$ uniquely as:

$$y = y_W + y_{W\perp}$$

with $y_W$ in $W$ and $y_{W\perp}$ in $W\perp$.

Moreover, if $B = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

This $y_W$ is $\text{proj}_W(y)$.

**Problem.** Use the formula to project $(1, 2, 3)$ to the $xy$-plane.
Orthogonal projection

Projecting onto any subspace

If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

This $y_W$ is $\text{proj}_W(y)$.

Problem. Let $W = \text{Span} \left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ and $y = e_1$. Find $y_W$. 
Orthogonal projection

Matrices for projections

Find $A$ so that $T(v) = Av$ is orthogonal projection onto

$$W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$
Suppose \( T(v) = Av \) is orthogonal projection onto a plane in \( \mathbb{R}^3 \). What is \( A^2 \) equal to?

1. \( A \)
2. \( A^{-1} \)
3. \( -A \)
4. \( 0 \)
5. \( I_n \)
6. \( -I_n \)

While you are at it: What are the eigenvalues of \( A \)?
Orthogonal bases
Finding coordinates with respect to orthogonal bases

Fact. Say that $B = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ and say that $y$ is in $W$. Then

$$y = c_1 u_1 + \cdots + c_k u_k$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

Same formula as before! Why?
Orthogonal bases

Finding coordinates with respect to orthogonal bases

**Fact.** Say that \( B = \{ u_1, \ldots, u_k \} \) is an orthogonal basis for a subspace \( W \) of \( \mathbb{R}^n \) and say that \( y \) is in \( W \). Then

\[
y = c_1 u_1 + \cdots + c_k u_k
\]

where

\[
c_i = \frac{y \cdot u_i}{u_i \cdot u_i}
\]

**Problem.** Find the \( B \)-coordinates of \((6, 1)\) where

\[
B = \left\{ \left( \begin{array}{c} 1 \\ 2 \end{array} \right), \left( \begin{array}{c} -4 \\ 2 \end{array} \right) \right\}
\]
Orthogonal bases
Finding coordinates with respect to orthogonal bases

**Fact.** Say that $B = \{u_1, \ldots, u_k\}$ is an orthogonal basis for a subspace $W$ of $\mathbb{R}^n$ and say that $y$ is in $W$. Then

$$y = c_1 u_1 + \cdots + c_k u_k$$

where

$$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

**Problem.** Find the $B$-coordinates of $(6, 1, -8)$ where

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$
Best approximation

\( W = \text{subspace of } \mathbb{R}^n \)

**Fact.** The projection \( y_W \) is the point in \( W \) closest to \( y \). In other words:

\[ \|y - y_W\| < \|y - w\| \]

for any \( w \) in \( W \) other than \( y_W \).

Why? Make a right triangle between \( y, y - y_W, \) and \( y_W - w \).
Best approximation

Problem. Find the distance from $e_1$ to

$$W = \text{Span} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$
Summary

- \( \text{proj}_W(y) = \text{orthogonal projection to } W \text{ of } y \)
- If \( B = \{u_1, \ldots, u_k\} \) is an orthogonal basis for \( W \) then
  \[
  y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k
  \]
  This \( y_W \) is \( \text{proj}_W(y) \).
- We find the matrix for projections in the usual way (project the \( e_i \)).
- If \( y \) is already in \( W \) then this gives the \( B \)-coordinates.
- The projection of \( y \) to \( W \) is the closest point in \( W \) to \( y \).