# Announcements: Nov 27

- CIOS open
- WebWork 6.1, 6.2, 6.3 due Wednesday
- WebWork 6.4 and 6.5 not due but on the final
- No quiz on Friday
- Final Exam on Tuesday Dec 12 6:00-8:50pm
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - CAS Study Session Dec 6 Clough 144/152

# Chapter 6

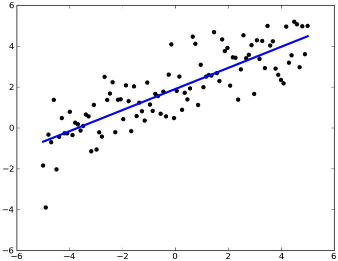
## Orthogonality and Least Squares

# Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can't solve $Ax = b$? How can we solve it as closely as possible?




The answer relies on orthogonality.

# Outline

- Orthogonal complements
- Computing orthogonal projections via orthogonal bases
- Orthogonal projections give closest points
- The Gram–Schmidt process: turn any basis into an orthogonal one

# Section 6.1

## Orthogonal Complements

# Orthogonal complements

$W$ = subspace of $\mathbb{R}^n$
$W^\perp = \{v \text{ in } \mathbb{R}^n \mid v \perp w \text{ for all } w \text{ in } W\}$

Theorem. $A = m \times n$ matrix

$$(\text{Row}A)^\perp = \text{Nul}\,A$$

Fact. Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $y$ in $\mathbb{R}^n$ can be written uniquely as

$$y_W + y_{W^\perp}$$

where $y_W \in W$ and $y_{W^\perp} \in W^\perp$.

Demo

# Section 6.2/6.3

## Orthogonal Projections

# Orthogonal Projections

Let $W$ be a subspace of $\mathbb{R}^n$ and $y$ a vector in $\mathbb{R}^n$.

$$\mathrm{proj}_W(y) = \text{orthogonal projection to } W \text{ of } y$$

Demo

If we write $y$ as $y_W + y_{W^\perp}$ then $\mathrm{proj}_W(y) = y_W$.

# Orthogonal projection as a linear transformation

Let $W$ be a subspace of $\mathbb{R}^n$.

We can think of orthogonal projection to $W$ as a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

The range of $T$ is $W$.

The null space of $T$ is $W^\perp$.

If $v$ is in $W$ then $T(v) = v$.

# Orthogonal projection

**Poll**

Suppose $T(v) = Av$ is orthogonal projection onto a plane in $\mathbb{R}^3$. What is $A^2$ equal to?

1. $A$
2. $A^{-1}$
3. $-A$
4. $0$
5. $I_n$
6. $-I_n$

While you are at it: What are the eigenvalues of $A$?

# Orthogonal projection onto a line

Say $W = \mathrm{Span}\{u\}$.

**Fact.** $\mathrm{proj}_W(y) = \frac{y \cdot u}{u \cdot u} u$

Demo

# Orthogonal projection

## Projecting onto any subspace

**Fact.** If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

**Fact.** If $y$ is in $W$ then this formula gives the $\mathcal{B}$-coordinates.

Demo

Demo

# Orthogonal bases

## Finding coordinates with respect to orthogonal bases

Fact. If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

Problem. Say that

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and say that $W$ is the span of $B$. Let $y = (6, 1, -8)$. Find $y_W$ and the $B$-coordinates of $y_W$.

# Best approximation

If we write $y$ as $y_W + y_{W^\perp}$ then $\text{proj}_W(y) = y_W$.

This point $\text{proj}_W(y) = y_W$ is the closest point in $W$ to $y$.

Demo

Demo

# Section 6.4

## The Gram–Schmidt Process

# Gram–Schmidt Process

## With two vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

# Gram–Schmidt Process

## With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

# Gram-Schmidt Process

## Example

Theorem. Say $\{u_1, \ldots, u_k\}$ is a basis for a nonzero subspace of $\mathbb{R}^n$. Define:

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= u_2 - \mathrm{proj}_{\mathrm{Span}\{v_1\}(u_2)} \\ v_3 &= u_3 - \mathrm{proj}_{\mathrm{Span}\{v_1,v_2\}(u_3)} \\ &\vdots \\ v_k &= u_k - \mathrm{proj}_{\mathrm{Span}\{v_1,\ldots,v_{k-1}\}(u_k)} \end{aligned}$$

Then $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $\mathrm{Span}\{u_1, \ldots, u_k\}$.

In other words, if at some stage you find a vector that is not orthogonal to the previous ones, then make it so!

# Gram–Schmidt Process

## With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

# Summary

- $\mathrm{proj}_W(y) =$ orthogonal projection to $W$ of $y$
- If $\mathcal{B} = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

  This $y_W$ is $\mathrm{proj}_W(y)$.
- We find the matrix for projections in the usual way (project the $e_i$).
- If $y$ is already in $W$ then this gives the $\mathcal{B}$-coordinates.
- The projection of $y$ to $W$ is the closest point in $W$ to $y$.
- To find an orthogonal basis, use Gram–Schmidt:

$$v_k = u_k - \mathrm{proj}_{\mathrm{Span}\{v_1,\ldots,v_{k-1}\}(u_k)}$$