Announcements: Nov 27

- CIOS open
- WebWork 6.1, 6.2, 6.3 due Wednesday
- WebWork 6.4 and 6.5 not due but on the final
- No quiz on Friday
- Final Exam on Tuesday Dec 12 6:00-8:50pm
- Upcoming Office Hours
  - Me: Monday 1-2 and Wednesday 3-4, Skiles 234
  - Bharat: Tuesday 1:45-2:45, Skiles 230
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

Other help:
- Math Lab, Clough 280, Mon - Thu 12-6
- Tutoring: http://www.successprograms.gatech.edu/tutoring
- CAS Study Session Dec 6 Clough 144/152
Chapter 6
Orthogonality and Least Squares
Where are we?

We have learned to solve $Ax = b$ and $Av = \lambda v$.

We have one more main goal.

What if we can’t solve $Ax = b$? How can we solve it as closely as possible?

The answer relies on orthogonality.
Outline

- Orthogonal complements
- Computing orthogonal projections via orthogonal bases
- Orthogonal projections give closest points
- The Gram–Schmidt process: turn any basis into an orthogonal one
Section 6.1
Orthogonal Complements
Orthogonal complements

$W = \text{subspace of } \mathbb{R}^n$

$W^\perp = \{ v \in \mathbb{R}^n \mid v \perp w \text{ for all } w \in W \}$

**Theorem.** $A = m \times n$ matrix

$$(\text{Row } A)^\perp = \text{Nul } A$$

**Fact.** Say $W$ is a subspace of $\mathbb{R}^n$. Then any vector $y$ in $\mathbb{R}^n$ can be written uniquely as

$$y = y_W + y_W^\perp$$

where $y_W \in W$ and $y_W^\perp \in W^\perp$. 

▶ Demo
Section 6.2/6.3
Orthogonal Projections
Orthogonal Projections

Let $W$ be a subspace of $\mathbb{R}^n$ and $y$ a vector in $\mathbb{R}^n$.

$$\text{proj}_W(y) = \text{orthogonal projection to } W \text{ of } y$$

If we write $y$ as $y_W + y_{W\perp}$ then $\text{proj}_W(y) = y_W$. 
Orthogonal projection as a linear transformation

Let $W$ be a subspace of $\mathbb{R}^n$.

We can think of orthogonal projection to $W$ as a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$.

The range of $T$ is $W$.

The null space of $T$ is $W^\perp$.

If $v$ is in $W$ then $T(v) = v$. 
Orthogonal projection

Suppose $T(v) = Av$ is orthogonal projection onto a plane in $\mathbb{R}^3$. What is $A^2$ equal to?

1. $A$
2. $A^{-1}$
3. $-A$
4. $0$
5. $I_n$
6. $-I_n$

While you are at it: What are the eigenvalues of $A$?
Orthogonal projection onto a line

Say $W = \text{Span}\{u\}$.

Fact. $\text{proj}_W(y) = \frac{y \cdot u}{u \cdot u} u$
Orthogonal projection

Projecting onto any subspace

Fact. If $B = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

Fact. If $y$ is in $W$ then this formula gives the $B$-coordinates.
Orthogonal bases
Finding coordinates with respect to orthogonal bases

Fact. If $B = \{u_1, \ldots, u_k\}$ is an orthogonal basis for $W$ then

$$y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k$$

Problem. Say that

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \right\}$$

and say that $W$ is the span of $B$. Let $y = (6, 1, -8)$. Find $y_W$ and the $B$-coordinates of $y_W$. 


Best approximation

If we write $y$ as $y_W + y_{W\perp}$ then $\text{proj}_W(y) = y_W$.

This point $\text{proj}_W(y) = y_W$ is the closest point in $W$ to $y$. 
Section 6.4
The Gram–Schmidt Process
Gram–Schmidt Process

With two vectors

Find an orthogonal basis for $W = \text{Span}\{u_1, u_2\}$, where

\[
\begin{align*}
    u_1 &= \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, &
    u_2 &= \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}
\end{align*}
\]
Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \text{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$
Theorem. Say \( \{u_1, \ldots, u_k\} \) is a basis for a nonzero subspace of \( \mathbb{R}^n \). Define:

\[
\begin{align*}
v_1 &= u_1 \\
v_2 &= u_2 - \text{proj}_{\text{Span}\{v_1\}}(u_2) \\
v_3 &= u_3 - \text{proj}_{\text{Span}\{v_1, v_2\}}(u_3) \\
\vdots \\
v_k &= u_k - \text{proj}_{\text{Span}\{v_1, \ldots, v_{k-1}\}}(u_k)
\end{align*}
\]

Then \( \{v_1, \ldots, v_k\} \) is an orthogonal basis for \( \text{Span}\{u_1, \ldots, u_k\} \).

In other words, if at some stage you find a vector that is not orthogonal to the previous ones, then make it so!
Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \text{Span}\{u_1, u_2, u_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \end{pmatrix}$$
Summary

- \( \text{proj}_W(y) = \) orthogonal projection to \( W \) of \( y \)
- If \( \mathcal{B} = \{u_1, \ldots, u_k\} \) is an orthogonal basis for \( W \) then
  \[
  y_W = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_k}{u_k \cdot u_k} u_k
  \]
  This \( y_W \) is \( \text{proj}_W(y) \).
- We find the matrix for projections in the usual way (project the \( e_i \)).
- If \( y \) is already in \( W \) then this gives the \( \mathcal{B} \)-coordinates.
- The projection of \( y \) to \( W \) is the closest point in \( W \) to \( y \).
- To find an orthogonal basis, use Gram–Schmidt:
  \[
  v_k = u_k - \text{proj}_{\text{Span}\{v_1, \ldots, v_{k-1}\}}(u_k)
  \]