Announcements: Nov 27

- Submit questions on Piazza for last day of class extravaganza
- CIOS open
- WebWork 6.1, 6.2, 6.3 due Wednesday
- WebWork 6.4 and 6.5 not due but on the final
- No quiz on Friday
- Final Exam on Tuesday Dec 12 6:00-8:50pm
- Upcoming Office Hours
  - Me: Wednesday 3-4, Skiles 234
  - Qianli: Wednesday 1-2, Clough 280
  - Arjun: Wednesday, 2:30-3:30, Skiles 230
  - Kemi: Thursday 9:30-10:30, Skiles 230
  - Martin: Friday 2-3, Skiles 230

  Other help:
  - Math Lab, Clough 280, Mon - Thu 12-6
  - Tutoring: http://www.successprograms.gatech.edu/tutoring
  - CAS Study Session Dec 6 Clough 144/152

Section 6.4

The Gram–Schmidt Process

Gram-Schmidt Process

Example

Theorem. Say $\{u_1, \ldots, u_k\}$ is a basis for a nonzero subspace of $\mathbb{R}^n$. Define:

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= u_2 - \mathrm{proj}_{\mathrm{Span}\{v_1\}}(u_2) \\ v_3 &= u_3 - \mathrm{proj}_{\mathrm{Span}\{v_1, v_2\}}(u_3) \\ &\vdots \\ v_k &= u_k - \mathrm{proj}_{\mathrm{Span}\{v_1, \ldots, v_{k-1}\}}(u_k) \end{aligned}$$

Then $\{v_1, \ldots, v_k\}$ is an orthogonal basis for $\mathrm{Span}\{u_1, \ldots, u_k\}$.

In other words, if at some stage you find a vector that is not orthogonal to the previous ones, then make it so!

Gram–Schmidt Process

With three vectors

Find an orthogonal basis for $W = \mathrm{Span}\{u_1, u_2, u_3\}$, where

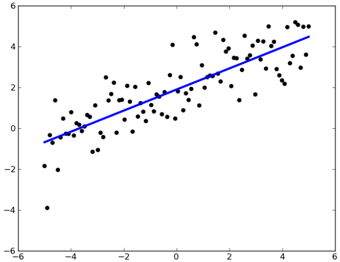
$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 4 \\ 4 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ -2 \\ 2 \\ 0 \end{pmatrix}$$

Section 6.5

Least Squares Problems

Least Squares problems

What if we can't solve $Ax = b$? How can we solve it as closely as possible?



To solve $Ax = b$ as closely as possible, we orthogonally project b onto $\mathrm{Col}(A)$; call the result $\widehat{b}$. Then solve $Ax = \widehat{b}$. This is the *least squares solution* to $Ax = b$.

Outline

- The method of least squares
- Application to best fit lines/planes
- Application to best fit curves

Least squares solutions

$A = m \times n$ matrix.

A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b.

Demo

Least squares solutions

A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b.

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

Why? By the Best Approximation Theorem, $A\widehat{x} = \mathrm{proj}_{\mathrm{Col}(A)}(b)$, so $b - A\widehat{x}$ is orthogonal to $\mathrm{Col}(A)$, so $b - A\widehat{x}$ is in the null space of A^T:

$$A^T(b - A\widehat{x}) = 0$$

Least squares solutions

A useful observation:

If we solve the least squares problem $Ax = b$ and a solution is $\widehat{x}$ then $A\widehat{x}$ is the projection of b onto the column space of A.

So we can use least squares to find projections instead of our previous formula.

Least squares solutions

Example

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

Find the least squares solutions to $Ax = b$ for the following A and b:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$$

What is the error?

Least squares solutions

Example

Theorem. The least squares solutions to $Ax = b$ are the solutions to

$$(A^T A)x = (A^T b)$$

Find the least squares solutions to $Ax = b$ for the following A and b:

$$A = \begin{pmatrix} 2 & 0 \\ -1 & 1 \\ 0 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

What is the error?

Least squares solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent:

1. $Ax = b$ has a unique least squares solution for all b in $\mathbb{R}^n$
2. The columns of A are linearly independent
3. $A^T A$ is invertible

In this case the least squares solution is $(A^T A)^{-1}(A^T b)$.

Application

Best fit lines

Problem. Find the best-fit line through $(0, 6)$, $(1, 0)$, and $(2, 0)$.

Demo

Best fit lines

Poll

What does the best fit line minimize?

1. the sum of the squares of the distances from the data points to the line
2. the sum of the squares of the vertical distances from the data points to the line
3. the sum of the squares of the horizontal distances from the data points to the line
4. the maximal distance from the data points to the line

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best fit circle/ellipse for the points:

$$(0,0),(2,0),(3,0),(0,1)$$

Gauss invented the method of least squares to predict the orbit of the asteroid Ceres as it passed behind the sun in 1801.

Demo

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best parabola (quadratic function of x) for the points:

$$(0, 0), (2, 0), (3, 0), (0, 1)$$

Demo

Least Squares Problems

More applications

Determine the least squares problem $Ax = b$ to find the best fit linear function $f(x, y)$

x	y	$f(x,y)$
1	0	0
0	1	1
-1	0	3
0	-1	4

Summary

- A least squares solution to $Ax = b$ is an $\widehat{x}$ in $\mathbb{R}^n$ so that $A\widehat{x}$ is as close as possible to b.
- Theorem. The least squares solutions to $Ax = b$ are the solutions to
$$(A^T A)x = (A^T b)$$
- To find a best fit line we plug our data points in for x and y in $Ax + By = C$. This gives a system of linear equations in A, B, C