Mathematics 1553
Midterm 2
Prof. Margalit

Section  G1/Arjun  G2/Talha  G3/Athreya  G4/Olivia  G5/James  (circle one!)
19 October 2018
1. Answer the following questions. No justification for your answer is required.

Suppose we have a set of 100 vectors in $\mathbb{R}^{99}$. Must it be true that the set is linearly dependent?

\begin{itemize}
  \item YES
  \item NO
\end{itemize}

Let $A$ be an $m \times n$ matrix and let $T(v) = Av$ be the associated linear transformation. Suppose that $T$ is not one-to-one. Must it be true that $Ax = 0$ has infinitely many solutions?

\begin{itemize}
  \item YES
  \item NO
\end{itemize}

Consider the function $T : \mathbb{R} \rightarrow \mathbb{R}$ given by $T(x) = x + 1$. Is $T$ a linear transformation?

\begin{itemize}
  \item YES
  \item NO
\end{itemize}

Suppose that $\{u, v\}$ is a basis for a subspace $V$ of $\mathbb{R}^3$. Must it be true that $\{u + v, v\}$ is a basis for $V$?

\begin{itemize}
  \item YES
  \item NO
\end{itemize}
2. Answer the following questions. No justification for your answer is required.

Complete the definition: A set of vectors \( \{v_1, \ldots, v_k\} \) in \( \mathbb{R}^n \) is linearly independent if...

the vector equation

\[
C_1v_1 + \cdots + C_kv_k = 0
\]

has only the trivial solution.

Consider the plane \( z = 1 \) in \( \mathbb{R}^3 \). Which properties of a subspace are failed by \( V \)? Select all that apply.

(a) Zero vector: the zero vector is in \( V \)
(b) Closure under addition: if \( u \) and \( v \) are in \( V \) then \( u + v \) is in \( V \)
(c) Closure under scalar multiplication: if \( u \) is in \( V \) and \( c \) is a scalar then \( cu \) is in \( V \)
(d) None of the above; \( V \) is a subspace

Write down a \( 2 \times 2 \) matrix so that the null space and column space both equal the line \( y = x \).

\[
\begin{pmatrix}
1 & -1 \\
1 & -1
\end{pmatrix}
\]

Write down a nonzero \( 2 \times 2 \) matrix \( A \) so that \( A \neq I \) and \( A^2 = A \).

\[
\begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}
\]
3. Consider the following matrix and its reduced row echelon form:

\[ A = \begin{pmatrix}
  2 & 4 & 1 & 11 & 0 \\
  3 & 6 & 1 & 16 & 0 \\
  7 & 14 & 3 & 38 & 0 \\
\end{pmatrix} \sim \begin{pmatrix}
  1 & 2 & 0 & 5 & 0 \\
  0 & 0 & 1 & 1 & 0 \\
  0 & 0 & 0 & 0 & 0 \\
\end{pmatrix} \]

Find a basis for \( \text{Col}(A) \).

\[ \left\{ \begin{pmatrix} 2 \\ 3 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\} \]

What is the dimension of \( \text{Col}(A) \)? 2

Find a basis for \( \text{Nul}(A) \).

\[ \begin{align*}
  x_1 &= -2x_2 - 5x_4 \\
  x_3 &= -x_4 \\
  x_2, x_4, x_5 & \text{ free.}
\end{align*} \]

\[ \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -5 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\} \]

What is the dimension of \( \text{Nul}(A) \)? 3

Is it possible for a \( 3 \times 5 \) matrix to have the dimensions of its column space and null space be equal?

YES  NO
4. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects over the line $y = -x$. What is the standard matrix for $T$?

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix}
\]

Consider the linear transformation $U : \mathbb{R}^3 \to \mathbb{R}^2$ given by the formula

\[
U \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ -x \end{pmatrix}
\]

What is the standard matrix for $U$?

\[
\begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]

Is $U$ one-to-one?  YES  NO

Draw the range of $U$.

What is the standard matrix for $T \circ U$?

\[
\begin{pmatrix}
0 & -1 \\
-1 & 0
\end{pmatrix} \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 \\
-1 & 0 & 0
\end{pmatrix}
\]
5. Consider the set of vectors

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ h \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix} \right\}
\]

For which values of \( h \) is the set linearly dependent?

\[
\begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & h \\ 1 & 1 & 3 \\ 1 & 1 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 3 \\ 0 & 1 & h \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & +2 \\ 0 & -3 & h-5 \end{pmatrix}
\]

\[
\sim \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & h+1 \end{pmatrix}
\]

\[
h = -1
\]

6. Find a basis for the plane in \( \mathbb{R}^3 \) defined by the equation \( x + 2y + z = 0 \).

\[
\begin{pmatrix} 1 & 1 & 1 \\ 2 & -2y & -z \\ 1 & y & z \\ 0 & 0 & 0 \end{pmatrix}
\]

\[
\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}
\]