Name SOLUTIONS

Mathematics 1553

Midterm 3

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1. Answer the following questions. No justification is required. Your four best scores on this page will count towards your exam grade.

Suppose that A is a 4×4 matrix whose only eigenvalue is 0. Suppose that the null space of A is a two-dimensional plane. Is A diagonalizable?

YES NO MAYBE

Suppose A is a 4×4 matrix with characteristic polynomial $x^4 + x^2 + x$. Is A invertible?

YES NO MAYBE

Suppose A is a 3×3 matrix with eigenvalues 1 and i. Is -1 is an eigenvalue of A?

YES NO MAYBE

Suppose that A and B are invertible matrices. Is $ABA^{-1}B^{-1}$ invertible?

YES NO MAYBE

Say A is a square matrix and $A(e_1 - e_2) = 0$. Is the transformation $T(v) = Av$ onto?

YES NO MAYBE

2. Answer the following questions. No justification is required. Your four best scores on this page will count towards your exam grade.

Complete the following definition: *A vector v is an eigenvector of a matrix A if...*

$v \neq 0$ & Av is a scalar multiple of v

Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is reflection about the plane $y = x$. What are the eigenvalues of the standard matrix for T?

$1, -1$

Find A^{100} if $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$. Give a simplified 2×2 matrix.

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Find the area of a triangle in $\mathbb{R}^2$ with vertices $(3, 4)$, $(13, 5)$, and $(103, 13)$.

$\frac{1}{2} \left| \det \begin{pmatrix} 10 & 100 \\ 1 & 9 \end{pmatrix} \right| = 5$

Compute the determinant of the following matrix.

$$\begin{pmatrix} 1 & 4 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

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3. Consider the following matrix:

$$A = \begin{pmatrix} 2 & 5 & 0 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

Find the eigenvalues of A.

1, 2

Find a basis for the eigenspace of the smaller eigenvalue of A.

$$\begin{pmatrix} 1 & 5 & 0 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x = -5y \\ y = y \\ z = 0 \end{matrix} \rightsquigarrow \left\{ \begin{pmatrix} -5 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Find a basis for the eigenspace of the larger eigenvalue of A.

$$\begin{pmatrix} 0 & 5 & 0 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightsquigarrow \begin{matrix} x = x \\ y = 0 \\ z = 0 \end{matrix} \rightsquigarrow \{e_1\}$$

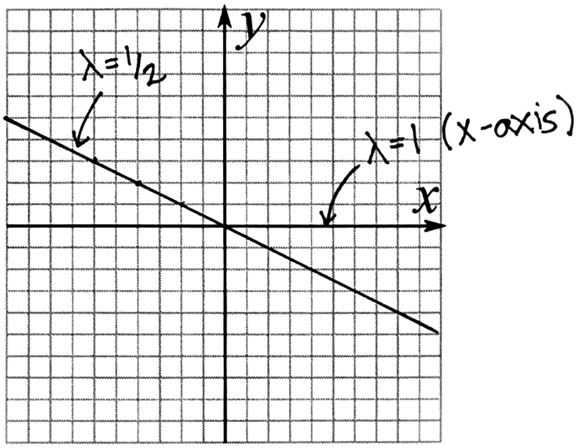
If A is diagonalizable, write a diagonalization $A = CDC^{-1}$. If not, explain why not.

It is not. The alg. mult. of 2 is 2,
but the dim. of the 2-eigenspace is only 1.

4. *(a)* Consider the following matrix.

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1/2 \end{pmatrix}$$

Draw the eigenspaces of A and label each eigenspace with the corresponding eigenvalue.



$\underline{\lambda=1}$ $\begin{pmatrix} 0 & 1 \\ 0 & -1/2 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \{e_1\}$

$\underline{\lambda=1/2}$ $\begin{pmatrix} 1/2 & 1 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \frac{1}{2}x + y = 0$

$\rightsquigarrow y = -\frac{1}{2}x$

Find a nonzero vector v so that the sequence of vectors $v, Av, A^2v, A^3v\ldots$ gets closer and closer to the origin.

$$\begin{pmatrix} -2 \\ 1 \end{pmatrix}$$

(b) Write down a 2×2 matrix that has eigenvalues 1 and -1 and corresponding eigenspaces equal to the x-axis and the line $y = x$ (in that order). Your answer should be completely simplified.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}$$

5. Compute the eigenvalues of the following matrix.

$$\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$$

$$\det \begin{pmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5$$

$$\rightsquigarrow \lambda = \frac{4 \pm \sqrt{-4}}{2} = 2 \pm i$$

Find an eigenvector for the eigenvalue with positive imaginary part.

$$\underline{\lambda = 2+i} \quad \begin{pmatrix} -1-i & -2 \\ 0 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$$

Find an eigenvector for the other eigenvalue. $\begin{pmatrix} -2 \\ 1-i \end{pmatrix}$

Which kinds of linear transformations of $\mathbb{R}^2$ have standard matrices without real eigenvalues?

(a) projections

(b) reflections

(c) rotations

(d) scalings