Section 3.2

Vector Equations and Spans

Outline of Section 3.2

- Learn the equivalences:

  vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems

- Learn the definition of span
- Learn the relationship between spans and consistency

Linear combinations, vector equations, and linear systems

We just saw the following question:

Is $\begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$ a linear combination of $\begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix}$?

And saw it was the same as a vector equation:

$$c_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}$$

which is the same as the system of linear equations:

$$\begin{aligned} c_1 - c_2 &= 8 \\ 2c_1 - 2c_2 &= 16 \\ 6c_1 - c_2 &= 3 \end{aligned}$$

which we solve by row reducing, and we get $(c_1, c_2) = (-1, -9)$.

Linear combinations, vector equations, and linear systems

In general, asking:

Is b a linear combination of $v_1, \ldots, v_k$?

is the same as asking if the vector equation

$$c_1 v_1 + \cdots c_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).

Span

Essential vocabulary word!

$$\begin{aligned}\mathrm{Span}\{v_1, v_2, \ldots, v_k\} &= \{c_1v_1 + c_2v_2 + \cdots c_kv_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)} \\ &= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k \\ &= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\end{aligned}$$

What are the possibilities for the span of two vectors in $\mathbb{R}^2$?

Demo

What are the possibilities for the span of three vectors in $\mathbb{R}^3$?

Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is at most the number of vectors you started with and is at most the dimension of the space they're in.

Span

Essential vocabulary word!

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Four ways of saying the same thing:

- b is in Span$\{v_1, v_2, \ldots, v_k\}$
- b is a linear combination of $v_1, \ldots, v_k$
- the vector equation $c_1v_1 + \cdots + c_kv_k = b$ has a solution
- the system of linear equations corresponding to

$$\left(\begin{array}{cccc|c} | & | & & | & | \\ v_1 & v_2 & \cdots & v_k & b \\ | & | & & | & | \end{array}\right),$$

  is consistent.

Demo Demo

Summary of Section 3.2

- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.
- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.