Section 3.2

Vector Equations and Spans
Outline of Section 3.2

- Learn the equivalences:
  
  vector equations $\leftrightarrow$ augmented matrices $\leftrightarrow$ linear systems

- Learn the definition of span

- Learn the relationship between spans and consistency
Linear combinations, vector equations, and linear systems

We just saw the following question:

Is \[
\begin{pmatrix}
8 \\
16 \\
3
\end{pmatrix}
\]
a linear combination of \[
\begin{pmatrix}
1 \\
2 \\
6
\end{pmatrix}
\]
and \[
\begin{pmatrix}
-1 \\
-2 \\
-1
\end{pmatrix}
\]?

And saw it was the same as a vector equation:

\[
c_1 \begin{pmatrix} 1 \\ 2 \\ 6 \end{pmatrix} + c_2 \begin{pmatrix} -1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 16 \\ 3 \end{pmatrix}
\]

which is the same as the system of linear equations:

\[
c_1 - c_2 = 8 \\
2c_1 - 2c_2 = 16 \\
6c_1 - c_2 = 3
\]

which we solve by row reducing, and we get \((c_1, c_2) = (-1, -9)\).
Linear combinations, vector equations, and linear systems

In general, asking:

Is $b$ a linear combination of $v_1, \ldots, v_k$?

is the same as asking if the vector equation

$$c_1 v_1 + \cdots + c_k v_k = b$$

is consistent, which is the same as asking if the system of linear equations corresponding to the augmented matrix

$$
\begin{pmatrix}
| & | & | \\
v_1 & v_2 & \cdots & v_k & | & | & b \\
| & | & |
\end{pmatrix},
$$

is consistent.

Compare with the previous slide! Make sure you are comfortable going back and forth between the specific case (last slide) and the general case (this slide).
Span

Essential vocabulary word!

\[
\text{Span}\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_i \text{ in } \mathbb{R}\} \leftarrow \text{(set builder notation)}
\]

\[= \text{the set of all linear combinations of vectors } v_1, v_2, \ldots, v_k\]
\[= \text{plane through the origin and } v_1, v_2, \ldots, v_k.\]

What are the possibilities for the span of two vectors in \(\mathbb{R}^2\)?

Demo

What are the possibilities for the span of three vectors in \(\mathbb{R}^3\)?

Demo

Conclusion: Spans are planes (of some dimension) through the origin, and the dimension of the plane is \textbf{at most} the number of vectors you started with and is \textbf{at most} the dimension of the space they’re in.
Span

Essential vocabulary word!

\[ \text{Span}\{v_1, v_2, \ldots, v_k\} = \{c_1 v_1 + c_2 v_2 + \cdots + c_k v_k \mid c_i \text{ in } \mathbb{R}\} \]

= the set of all linear combinations of vectors \(v_1, v_2, \ldots, v_k\)

= plane through the origin and \(v_1, v_2, \ldots, v_k\).

Four ways of saying the same thing:

- \(b\) is in \(\text{Span}\{v_1, v_2, \ldots, v_k\}\)
- \(b\) is a linear combination of \(v_1, \ldots, v_k\)
- the vector equation \(c_1 v_1 + \cdots + c_k v_k = b\) has a solution
- the system of linear equations corresponding to

\[
\begin{pmatrix}
| & | & \cdots & | & | \\
v_1 & v_2 & \cdots & v_k & b
\end{pmatrix},
\]

is consistent.
Summary of Section 3.2

- Checking if a linear system is consistent is the same as asking if the column vector on the end of an augmented matrix is in the span of the other column vectors.

- Spans are planes, and the dimension of the plane is at most the number of vectors you started with.