Announcements: Sep 10

- Midterm 1 on Sep 21
- Quiz 3 Friday in recitation
- WeBWorK 3.1 and 3.2 due Wednesday
- My office hours Wednesday 2:00-3:00 and Friday 9:30-10:30 in Skiles 234
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday - Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
- Supplemental problems on master course web site

Section 3.3

Matrix equations

Outline Section 3.3

- Understand the equivalences:

  linear system $\leftrightarrow$ augmented matrix $\leftrightarrow$ vector equation $\leftrightarrow$ matrix equation

- Understand the equivalence:

  $$Ax = b \text{ is consistent} \longleftrightarrow b \text{ is in the span of the columns of } A$$

  (also: what does this mean geometrically)
- Learn for which A the equation $Ax = b$ is always consistent
- Learn to multiply a vector by a matrix

Multiplying Matrices

$$\text{row vector} \times \text{column vector} : \begin{pmatrix} a_1 & \cdots & a_n \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + \cdots + a_n b_n$$

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} =$$

Multiplying Matrices

Another way to multiply

$$\text{matrix} \times \text{column vector} : \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

OR

$$\text{matrix} \times \text{col vector} : \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & \cdots & b_n c_n \\ | & & | \end{pmatrix}$$

Read this as: b_1 times the first column c_1 is the first column of the answer, b_2 times c_2 is the second column of the answer...

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 7 \\ 8 \end{pmatrix} =$$

Linear Systems vs Augmented Matrices vs Matrix Equations vs Vector Equations

A matrix equation is an equation $Ax = b$ where A is a matrix and b is a vector. So x is a vector of variables.

A is an $m \times n$ matrix if it has m rows and n columns.
What sizes must x and b be?

Example:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 10 \\ 11 \end{pmatrix}$$

Rewrite this equation as a vector equation, a system of linear equations, and an augmented matrix.

We will go back and forth between these four points of view over and over again. You need to get comfortable with this.

Solutions to Linear Systems vs Spans

Say that

$$A = \begin{pmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{pmatrix}.$$

Fact. $Ax = b$ has a solution $\iff b$ is in the span of columns of A

Why?

Again this is a basic fact we will use over and over and over.

Solutions to Linear Systems vs Spans

Fact. $Ax = b$ has a solution $\iff$ b is in the span of columns of A

Examples:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}$$

Is a given vector in the span?

Fact. $Ax = b$ has a solution $\iff b$ is in the span of columns of A

Is $(9, 10, 11)$ in the span of $(1, 3, 5)$ and $(2, 4, 6)$?

Is a given vector in the span?

Poll

Which of the following true statements can you verify without row reduction?

1. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,10,20)$, $(0,-1,-2)$
2. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,5,7)$, $(0,6,8)$
3. $(0,1,2)$ is in the span of $(3,3,4)$, $(0,1,0)$, $(0,0,\sqrt{2})$
4. $(0,1,2)$ is in the span of $(5,7,0)$, $(6,8,0)$, $(3,3,4)$

Pivots vs Solutions

Theorem. Let A be an $m \times n$ matrix. The following are equivalent.

1. $Ax = b$ has a solution for all b
2. The span of the columns of A is $\mathbb{R}^m$
3. A has a pivot in each row

Why?

More generally, if you have k vectors in $\mathbb{R}^n$ and you want to know the dimension of the span, you should row reduce and count the number of pivots.

Properties of the Matrix Product Ax

$c =$ real number, $u, v =$ vectors,

- $A(u+v) = Au + Av$
- $A(cv) = cAv$

Application. If u and v are solutions to $Ax = 0$ then so is every element of $\mathrm{Span}\{u+v\}$.

Guiding questions

Here are the guiding questions for the rest of the chapter:

1. What are the solutions to $Ax = 0$?

2. For which b is $Ax = b$ consistent?

These are two separate questions!

Summary of Section 3.3

- Two ways to multiply a matrix times a column vector:

$$\begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix} b = \begin{pmatrix} r_1 b \\ \vdots \\ r_m b \end{pmatrix}$$

OR

$$\begin{pmatrix} | & | & & | \\ c_1 & c_2 & \cdots & c_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} | & & | \\ b_1 c_1 & \cdots & b_n c_n \\ | & & | \end{pmatrix}$$

- Linear systems, augmented matrices, vector equations, and matrix equations are all equivalent.
- Fact. $Ax = b$ has a solution $\Leftrightarrow$ b is in the span of columns of A
- Theorem. Let A be an $m \times n$ matrix. The following are equivalent.
  1. $Ax = b$ has a solution for all b
  2. The span of the columns of A is $\mathbb{R}^m$
  3. A has a pivot in each row