Announcements: Sep 19

- Midterm 1 on Friday in recitation
- WeBWorK 3.3 and 3.4 due tonite!
- My office hours Thu 2:30-3:30 and Friday 9:30-10:30 in Skiles 234
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Thu 11:15-12:15 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 3:30-4:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
- Review Sessions
  - Talha Thu 7-8:30 Mason 1133 (doors close at 7!)
  - Arjun tba
- Supplemental problems and practice exams on master course web site
Section 3.6

Subspaces
Outline of Section 3.6

- Definition of subspace
- Examples and non-examples of subspaces
- Spoiler alert: Subspaces are the same as spans
- Spanning sets for subspaces
- Two important subspaces for a matrix: $\text{Col}(A)$ and $\text{Nul}(A)$
Subspaces

A subspace of \( \mathbb{R}^n \) is a subset \( V \) of \( \mathbb{R}^n \) with:

1. The zero vector is in \( V \).
2. If \( u \) and \( v \) are in \( V \), then \( u + v \) is also in \( V \).
3. If \( u \) is in \( V \) and \( c \) is a scalar, then \( cu \) is in \( V \).

The second and third properties are called “closure under addition” and “closure under scalar multiplication.”

Together, the second and third properties could together be rephrased as: closure under linear combinations.
Which are subspaces?

1. the unit circle in $\mathbb{R}^2$

2. the point $(1, 2, 3)$ in $\mathbb{R}^3$

3. the $xy$-plane in $\mathbb{R}^3$

4. the $xy$-plane together with the $z$-axis in $\mathbb{R}^3$
Which are subspaces?

Poll

Is the first quadrant of $\mathbb{R}^2$ a subspace?

1. yes
2. no
Which are subspaces?

1. \( \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a + b = 0 \right\} \)

2. \( \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a + b = 1 \right\} \)

3. \( \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid ab \neq 0 \right\} \)

4. \( \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mid a, b \text{ rational} \right\} \)
Spans and subspaces

**Fact.** Any $\text{Span}\{v_1, \ldots, v_k\}$ is a subspace.

Why?

**Fact.** Every subspace $V$ is a span.

Why?

So now we know that three things are the same:

- subspaces
- spans
- planes through 0

So why bother with the word “subspace”? Sometimes easier to check a subset is a subspace than to check it is a span (see null spaces, eigenspaces). Also, it makes sense (and is often useful) to think of a subspace *without a particular spanning set in mind.*
Column Space and Null Space

\[ A = m \times n \text{ matrix.} \]

\[ \text{Col}(A) = \text{column space of } A = \text{span of the columns of } A \]

\[ \text{Nul}(A) = \text{null space of } A = (\text{set of solutions to } Ax = 0) \]

Example. \[ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \]

\[ \text{Col}(A) = \text{subspace of } \mathbb{R}^m \]

\[ \text{Nul}(A) = \text{subspace of } \mathbb{R}^n \]
Spanning sets for $\text{Nul}(A)$ and $\text{Col}(A)$

Find spanning sets for $\text{Nul}(A)$ and $\text{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$\text{Nul}(A)$ is the span of solutions to $Ax = 0$, which for this matrix we can show is given by

$$y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

So the basis for nullspace of $A$ is $$\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \}.$$
Spanning sets for \( \text{Nul}(A) \) and \( \text{Col}(A) \)

In general:

- our usual parametric solution for \( Ax = 0 \) gives a spanning set for \( \text{Nul}(A) \)
- the pivot columns of \( A \) form a spanning set for \( \text{Col}(A) \)

**Warning!** Not the pivot columns of the reduced matrix.

Notice that the columns of \( A \) form a (possibly larger) spanning set. We'll see later that the above recipe is the smallest spanning set.
Spanning sets

Find a spanning set for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$. 
Subspaces and Null spaces

Fact. Every subspace is a null space.

Why? Given a spanning set, you can reverse engineer the $A$...

So now we know that four things are the same:

- subspaces
- spans
- planes through 0
- solutions to $Ax = 0$
So why learn about subspaces?

If subspaces are the same as spans, planes through the origin, and solutions to $Ax = 0$, why bother with this new vocabulary word?

The point is that we have been throwing around terms like “3-dimensional plane in $\mathbb{R}^4$” all semester, but we never said what “dimension” and “plane” are. Subspaces give the proper way to define a plane. Soon we will learn the meaning of a dimension of a subspace.
Section 3.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.

- Two important subspaces: $\text{Nul}(A)$ and $\text{Col}(A)$

- Find a spanning set for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form

- Find a spanning set for $\text{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)

- Four things are the same: subspaces, spans, planes through 0, null spaces