Announcements: Sep 19

- Midterm 1 on Friday in recitation
- WeBWorK 3.3 and 3.4 due tonite!
- My office hours Thu 2:30-3:30 and Friday 9:30-10:30 in Skiles 234
- TA Office Hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Thu 11:15-12:15 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 3:30-4:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
- Review Sessions
  - Talha Thu 7-8:30 Mason 1133 (doors close at 7!)
  - Arjun tba
- Supplemental problems and practice exams on master course web site

Section 3.6

Subspaces

Outline of Section 3.6

- Definition of subspace
- Examples and non-examples of subspaces
- Spoiler alert: Subspaces are the same as spans
- Spanning sets for subspaces
- Two important subspaces for a matrix: $\mathrm{Col}(A)$ and $\mathrm{Nul}(A)$

Subspaces

A subspace of $\mathbb{R}^n$ is a subset V of $\mathbb{R}^n$ with:

1. The zero vector is in V.
2. If u and v are in V, then $u + v$ is also in V.
3. If u is in V and c is a scalar, then cu is in V.

The second and third properties are called "closure under addition" and "closure under scalar multiplication."

Together, the second and third properties could together be rephrased as: closure under linear combinations.

Which are subspaces?

1. the unit circle in $\mathbb{R}^2$
2. the point $(1, 2, 3)$ in $\mathbb{R}^3$
3. the xy-plane in $\mathbb{R}^3$
4. the xy-plane together with the z-axis in $\mathbb{R}^3$

Which are subspaces?

Poll

Is the first quadrant of $\mathbb{R}^2$ a subspace?

1. yes
2. no

Which are subspaces?

1. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a+b=0 \right\}$

2. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a+b=1 \right\}$

3. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, ab \neq 0 \right\}$

4. $\left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbb{R}^2 \,|\, a, b \text{ rational} \right\}$

Spans and subspaces

Fact. Any $\mathrm{Span}\{v_1, \ldots, v_k\}$ is a subspace.

Why?

Fact. Every subspace V is a span.

Why?

So now we know that three things are the same:

- subspaces
- spans
- planes through 0

So why bother with the word "subspace"? Sometimes easier to check a subset is a subspace than to check it is a span (see null spaces, eigenspaces). Also, it makes sense (and is often useful) to think of a subspace *without a particular spanning set in mind*.

Column Space and Null Space

$A = m \times n$ matrix.

$\text{Col}(A)$ = column space of A = span of the columns of A

$\text{Nul}(A)$ = null space of A = (set of solutions to $Ax = 0$)

Example. $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\text{Col}(A)$ = subspace of $\mathbb{R}^m$

$\text{Nul}(A)$ = subspace of $\mathbb{R}^n$

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

Find spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Spanning sets for $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$

In general:

- our usual parametric solution for $Ax = 0$ gives a spanning set for $\mathrm{Nul}(A)$
- the pivot columns of A form a spanning set for $\mathrm{Col}(A)$

Warning! Not the pivot columns of the reduced matrix.

Notice that the columns of A form a (possibly larger) spanning set. We'll see later that the above recipe is the smallest spanning set.

Spanning sets

Find a spanning set for the plane $2x + 3y + z = 0$ in $\mathbb{R}^3$.

Subspaces and Null spaces

Fact. Every subspace is a null space.

Why? Given a spanning set, you can reverse engineer the A...

So now we know that four things are the same:

- subspaces
- spans
- planes through 0
- solutions to $Ax = 0$

So why learn about subspaces?

If subspaces are the same as spans, planes through the origin, and solutions to $Ax = 0$, why bother with this new vocabulary word?

The point is that we have been throwing around terms like "3-dimensional plane in $\mathbb{R}^4$" all semester, but we never said what "dimension" and "plane" are. Subspaces give the proper way to define a plane. Soon we will learn the meaning of a dimension of a subspace.

Section 3.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset V with:
  1. The zero vector is in V.
  2. If u and v are in V, then $u + v$ is also in V.
  3. If u is in V and c is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of A (not reduced A)
- Four things are the same: subspaces, spans, planes through 0, null spaces