Announcements: October 10

- Midterm 2 Oct 19 in recitation
- Quiz on 4.2 & 4.3 Friday in recitation
- WeBWorK due Friday!
- My office hours Wed 2-3 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123
Section 4.4

Matrix Multiplication
Section 4.4 Outline

- Understand composition of linear transformations
- Learn how to multiply matrices
- Learn the connection between these two things
Function composition

Remember from calculus that if \( f \) and \( g \) are functions then the composition \( f \circ g \) is a new function defined as follows:

\[
f \circ g(x) = f(g(x))
\]

In words: first apply \( g \), then \( f \).

Example: \( f(x) = x^2 \) and \( g(x) = x + 1 \).

Note that \( f \circ g \) is usually different from \( g \circ f \).
Composition of linear transformations

We can do the same thing with linear transformations $T : \mathbb{R}^m \to \mathbb{R}^p$ and $U : \mathbb{R}^n \to \mathbb{R}^m$ and make the composition $T \circ U$.

Notice that both have an $m$. Why?

What are the domain and codomain for $T \circ U$?

Associative property: $(S \circ T) \circ U = S \circ (T \circ U)$

Why?
Composition of linear transformations

Example. $T =$ projection to $y$-axis and $U =$ reflection about $y = x$ in $\mathbb{R}^2$

What is the standard matrix for $T \circ U$?

What about $U \circ T$?
Matrix Multiplication
And now for something completely different (not really!)

Suppose \( A \) is an \( m \times n \) matrix. We write \( a_{ij} \) or \( A_{ij} \) for the \( ij \)th entry.

If \( A \) is \( m \times n \) and \( B \) is \( n \times p \), then \( AB \) is \( m \times p \) and

\[
(AB)_{ij} = r_i \cdot b_j
\]

where \( r_i \) is the \( i \)th row of \( A \), and \( b_j \) is the \( j \)th column of \( B \).

Or: the \( j \)th column of \( AB \) is \( A \) times the \( j \)th column of \( B \).

Multiply these matrices (both ways):

\[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
0 & -2 \\
1 & -1 \\
2 & 0
\end{pmatrix}
\]
Matrix Multiplication and Linear Transformations

As above, the composition $T \circ U$ means: do $U$ then do $T$

**Fact.** The standard matrix for a composition of linear transformations is the product of the standard matrices.

Why? Say that the standard matrices for $T$ and $U$ are $A$ and $B$:

$$(T \circ U)(v) = T(U(v)) = T(Bv) = A(Bv)$$

So we need to check that $A(Bv) = (AB)v$. Enough to do this for $v = e_i$. In this case $Bv$ is the $i$th column of $B$. So the left-hand side is $A$ times the $i$th column of $B$. The right-hand side is the $i$th column of $AB$ which we already said was $A$ times the $i$th column of $B$. It works!
Matrix Multiplication and Linear Transformations

Fact. The matrix for a composition of linear transformations is the product of the standard matrices.

Example. $T = \text{projection to } y\text{-axis}$ and $U = \text{reflection about } y = x \text{ in } \mathbb{R}^2$

What is the standard matrix for $T \circ U$?
Linear transformations are matrix transformations

Find the standard matrix for the linear transformation of \( \mathbb{R}^3 \) that reflects through the \( xy \)-plane and then projects onto the \( yz \)-plane.
Discussion Question

Are there nonzero matrices $A$ and $B$ with $AB = 0$?

1. Yes
2. No
Properties of Matrix Multiplication

- \( A(BC) = (AB)C \)
- \( A(B + C) = AB + AC \)
- \( (B + C)A = BA + CA \)
- \( r(AB) = (rA)B = A(rB) \)
- \( (AB)^T = B^T A^T \)
- \( I_mA = A = AI_n \), where \( I_k \) is the \( k \times k \) identity matrix.

Multiplication is associative because function composition is (this would be hard to check from the definition!).

Warning!

- \( AB \) is not always equal to \( BA \)
- \( AB = AC \) does not mean that \( B = C \)
- \( AB = 0 \) does not mean that \( A \) or \( B \) is 0
Sums and Scalar Multiples

Same as for vectors: component-wise, so matrices must be same size to add.

\[ A + B = B + A \]

\[ (A + B) + C = A + (B + C) \]

\[ r(A + B) = rA + rB \]

\[ (r + s)A = rA + sA \]

\[ (rs)A = r(sA) \]

\[ A + 0 = A \]

(We can define linear transformations \( T + U \) ad \( cT \), and so all of the above facts are also facts about linear transformations.)
Summary of Section 4.4

- Composition: \((T \circ U)(v) = T(U(v))\)  (do \(U\) then \(T\))
- Matrix multiplication: \((AB)_{ij} = r_i \cdot b_j\)
- Matrix multiplication: the \(i\)th column of \(AB\) is \(A(b_i)\)
- The standard matrix for a composition of linear transformations is the product of the standard matrices.

**Warning!**

- \(AB\) is not always equal to \(BA\)
- \(AB = AC\) does not mean that \(B = C\)
- \(AB = 0\) does not mean that \(A\) or \(B\) is 0