Announcements: October 15

- Midterm 2 Friday in recitation, §3.5–4.4
- WeBWorK due Wednesday
- Supplemental problems and Practice exam on web site
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123

Section 4.5

Matrix Inverses

Section 4.5 Outline

- The definition of a matrix inverse
- How to find a matrix inverse
- Inverses for linear transformations

Inverses

$A = n \times n$ matrix.

A is invertible if there is a matrix B with

$$AB = BA = I_n$$

B is called the inverse of A and is written A^{-1}

Example:

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

The 2×2 Case

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then $\det(A) = ad - bc$ is the determinant of A.

Fact. If $\det(A) \neq 0$ then A is invertible and $A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

If $\det(A) = 0$ then A is not invertible.

Example. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

Solving Linear Systems via Inverses

Fact. If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

Solve

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 1 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

Solving Linear Systems via Inverses

What if we change b?

$$\begin{aligned} 2x + 3y + 2z &= 1 \\ x + 3z &= 0 \\ 2x + 2y + 3z &= 1 \end{aligned}$$

Using

$$\begin{pmatrix} 2 & 3 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 3 \end{pmatrix}^{-1} = \begin{pmatrix} -6 & -5 & 9 \\ 3 & 2 & -4 \\ 2 & 2 & -3 \end{pmatrix}$$

So finding the inverse is essentially the same as solving all $Ax = b$ equations at once (fixed A, varying b).

Some Facts

Say that A and B are invertible $n \times n$ matrices.

- A^{-1} is invertible and $(A^{-1})^{-1} = A$
- AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

What is $(ABC)^{-1}$?

A recipe for the inverse

Suppose $A = n \times n$ matrix.

- Row reduce $(A \mid I_n)$
- If reduction has form $(I_n \mid B)$ then A is invertible and $B = A^{-1}$.
- Otherwise, A is not invertible.

Example. Find $\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -3 & -4 \end{pmatrix}^{-1}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & -3 & -4 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 3 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -6 & -2 \\ 0 & 1 & 0 & 0 & -2 & -1 \\ 0 & 0 & 1 & 0 & 3/2 & 1/2 \end{array}\right)$$

Why Does This Work?

First answer: we can think of the algorithm as simultaneously solving

$$Ax_1 = e_1$$
$$Ax_2 = e_2$$

and so on. But the columns of A^{-1} are $A^{-1}e_i$, which is x_i.

The Invertible Matrix Theorem

Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.

(1) A is invertible
(2) T is invertible
(3) The reduced row echelon form of A is I_n
(4) A has n pivots
(5) $Ax = 0$ has only 0 solution
(6) $\mathrm{Nul}(A) = \{0\}$
(7) $\mathrm{nullity}(A) = 0$
(8) columns of A are linearly independent
(9) columns of A form a basis for $\mathbb{R}^n$
(10) T is one-to-one
(11) $Ax = b$ is consistent for all b in $\mathbb{R}^n$
(12) $Ax = b$ has a unique solution for all b in $\mathbb{R}^n$
(13) columns of A span $\mathbb{R}^n$
(14) $\mathrm{Col}(A) = \mathbb{R}^n$
(15) $\mathrm{rank}(A) = n$
(16) T is onto
(17) A has a left inverse
(18) A has a right inverse

The Invertible Matrix Theorem

There are two kinds of square matrices, invertible and non-invertible matrices.

For invertible matrices, all of the conditions in the IMT hold. And for a non-invertible matrix, all of them fail to hold.

Example

Determine whether A is invertible. $A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix}$

It isn't necessary to find the inverse. Instead, we may use the Invertible Matrix Theorem by checking whether we can row reduce to obtain three pivot columns, or three pivot positions.

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \end{pmatrix}$$

There are three pivot positions, so A is invertible by the IMT (statement c).

The Invertible Matrix Theorem

Poll

Which are true? Why?

m) If A is invertible then the rows of A span $\mathbb{R}^n$
n) If $Ax = b$ has exactly one solution for all b in $\mathbb{R}^n$ then A is row equivalent to the identity.
o) If A is invertible then A^2 is invertible
p) If A^2 is invertible then A is invertible

Invertible Functions

A function $T : \mathbb{R}^n \to \mathbb{R}^n$ is **invertible** if there is a function $U : \mathbb{R}^n \to \mathbb{R}^n$, so

$$T \circ U = U \circ T = \text{identity}$$

That is,

$$T \circ U(v) = U \circ T(v) = v \text{ for all } v \in \mathbb{R}^n$$

Fact. Suppose $A = n \times n$ matrix and T is the matrix transformation. Then T is invertible *as a function* if and only if A is invertible. And in this case, the standard matrix for T^{-1} is A^{-1}.

Example. Counterclockwise rotation by $\pi/2$.

Summary of Section 4.5

- A is invertible if there is a matrix B (called the inverse) with

$$AB = BA = I_n$$

- For a 2×2 matrix A we have that A is invertible exactly when $\det(A) \neq 0$ and in this case

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

- If A is invertible, then $Ax = b$ has exactly one solution:

$$x = A^{-1}b.$$

- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \mid I_n)$.
- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  (1) A is invertible
  (2) T is invertible
  (3) The reduced row echelon form of A is I_n
  (4) etc.