Announcements: October 29

- Come pick up your midterm up front.
- Midterm 3 on §4.5-6.5 Nov 16 in recitation
- Quiz 5.1-5.3 Friday in recitation
- WeBWorK 5.1-5.3 due Wednesday
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Tue 11-12 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 11-12 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280 ▸ Schedule
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123

Where are we?

Remember:

> Almost every engineering problem, no matter how huge, can be reduced to linear algebra:
>
> $$Ax = b \quad \text{or}$$
> $$Ax = \lambda x$$

A few examples of the second: column buckling, control theory, image compression, exploring for oil, materials, natural frequency (bridges and car stereos), principal component analysis, Google, Netflix, and many more!

We have said most of what we are going to say about the first problem. We now begin in earnest on the second problem.

A Question from Biology

In a population of rabbits...

- half of the new born rabbits survive their first year
- of those, half survive their second year
- the maximum life span is three years
- rabbits produce 0, 6, 8 rabbits in their first, second, and third years

If I know the population one year - think of it as a vector (f, s, t) - what is the population the next year?

Now choose some starting population vector $u = (f, s, t)$ and choose some number of years N. What is the new population after N years?

Demo

Eigenvectors and Eigenvalues

Suppose A is an $n \times n$ matrix and there is a $v \neq 0$ in $\mathbb{R}^n$ and λ in $\mathbb{R}$ so that

$$Av = \lambda v$$

then v is called an eigenvector for A, and λ is the corresponding eigenvalue.

eigen = *characteristic*

So Av points in the same direction as v.

This the the most important definition in the course.

Demo

Eigenvectors and Eigenvalues

Examples

$$A = \begin{pmatrix} 0 & 6 & 8 \\ 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \end{pmatrix}, \quad v = \begin{pmatrix} 32 \\ 8 \\ 2 \end{pmatrix}, \quad \lambda = 2$$

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda = 4$$

How do you check?

Poll

Which of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are eigenvectors of

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}?$$

What are the eigenvalues?

Eigenvectors and Eigenvalues

Confirming eigenvalues

Confirm that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 2 & -4 \\ -1 & -1 \end{pmatrix}$.

What is a general procedure for finding eigenvalues?

Eigenspaces

Let A be an $n \times n$ matrix. The set of eigenvectors for a given eigenvalue λ of A (plus the zero vector) is a subspace of $\mathbb{R}^n$ called the λ-eigenspace of A.

Why is this a subspace?

Fact. λ-eigenspace for $A = \mathrm{Nul}(A - \lambda I)$

Example. Find the eigenspaces for $\lambda = 2$ and $\lambda = -1$ and sketch.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

Find a basis for the 2–eigenspace:

$$\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}$$

Eigenvalues

And invertibility

Fact. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A

Why?

Eigenvalues
Triangular matrices

Fact. The eigenvalues of a triangular matrix are the diagonal entries.

Why?

Eigenvalues

Distinct eigenvalues

Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.

Why?

Eigenvalues geometrically

If v is an eigenvector of A then that means v and Av are scalar multiples, i.e. they lie on a line.

Without doing any calculations, find the eigenvectors and eigenvalues of the matrices corresponding to the following linear transformations:

- Reflection about the line $y = -x$ in $\mathbb{R}^2$
- Orthogonal projection onto the x-axis in $\mathbb{R}^2$
- Rotation of $\mathbb{R}^2$ by $\pi/2$ (counterclockwise)
- Scaling of $\mathbb{R}^2$ by 3
- (Standard) shear of $\mathbb{R}^2$
- Orthogonal projection to the xy-plane in $\mathbb{R}^3$

Demo

Summary of Section 6.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- The λ-eigenspace of A is the solution to $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow$ 0 is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations