Announcements: November 5

- Midterm 3 on §4.5-6.5 Nov 16 in recitation
- Quiz 6.1-6.2 Friday in recitation.
- WeBWorK 6.1-6.2 due Wednesday
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Tue 11-12 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 11-12 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7
  - Mon/Wed 6-7
Eigenvalues in Structural Engineering

Watch this video about the Tacoma Narrows bridge. ➤ Watch

Here are some toy models. ➤ Check it out

The masses move the most at their natural frequencies \( \omega \). To find those, use the spring equation: \( mx'' = -kx \leftrightarrow \sin(\omega t) \).

With 3 springs and 2 equal masses, we get:

\[
mx''_1 = -kx_1 + k(x_2 - x_1) \\
mx''_2 = -kx_2 + k(x_1 - x_2)
\]

Guess a solution \( x_1(t) = A_1(\cos(\omega t) + i \sin(\omega t)) \) and similar for \( x_2 \). Finding \( \omega \) reduces to finding eigenvalues of \( \begin{pmatrix} -2k & k \\ k & -2k \end{pmatrix} \).

Eigenvectors: \((1, 1) \& (1, -1)\) (in/out of phase) ➤ Details
Section 6.4
Diagonalization
Section 6.4 Outline

• Diagonalization
• Using diagonalization to take powers
• Algebraic versus geometric dimension
We understand diagonal matrices

We completely understand what diagonal matrices do to $\mathbb{R}^n$. For example:

\[
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}
\]

We have seen that it is useful to take powers of matrices: for instance in computing rabbit populations.

If $A$ is diagonal, powers of $A$ are easy to compute. For example:

\[
\begin{pmatrix}
2 & 0 \\
0 & 3
\end{pmatrix}^{10} = \ldots
\]
Powers of matrices that are similar to diagonal ones

What if $A$ is not diagonal? Suppose want to understand the matrix

$$A = \begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix}$$

generically? Or take it's 10th power? What would we do?

What if I give you the following equality:

$$\begin{pmatrix} 5/4 & 3/4 \\ 3/4 & 5/4 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$A = C D C^{-1}$$

This is called diagonalization.

How does this help us understand $A$? Or find $A^{10}$?
Diagonalization

Suppose $A$ is $n \times n$. We say that $A$ is diagonalizable if we can write:

$$A = CDC^{-1} \quad D = \text{diagonal}$$

We say that $A$ is similar to $D$.

How does this factorization of $A$ help describe what $A$ does to $\mathbb{R}^n$? How does this help us take powers of $A$?

Understanding the rabbit example: since 2 is the largest eigenvalue, (almost) all other vectors get pulled towards that eigenvector. Compare with the example from the last slide.
Diagonalization

**Theorem.** A is diagonalizable \( \iff \) A has \( n \) linearly independent eigenvectors.

In this case

\[
A = \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix} \begin{pmatrix} v_1 & v_2 & \cdots & v_n \end{pmatrix}^{-1}
\]

\[
= C \quad D \quad C^{-1}
\]

where \( v_1, \ldots, v_n \) are linearly independent eigenvectors and \( \lambda_1, \ldots, \lambda_n \) are the corresponding eigenvalues (in order).

Why?
Example

Diagonalize if possible.

\[
\begin{pmatrix}
2 & 6 \\
0 & -1
\end{pmatrix}
\]
Example

Diagonalize if possible.

\[
\begin{pmatrix}
3 & 1 \\
0 & 3
\end{pmatrix}
\]
Example

Diagonalize if possible.

\[
\begin{pmatrix}
\frac{3}{4} & \frac{1}{4} \\
\frac{1}{4} & \frac{3}{4}
\end{pmatrix}
\]
More Examples

Diagonalize if possible.

\[
\begin{pmatrix}
0 & 1 \\
1 & 1
\end{pmatrix},
\begin{pmatrix}
a & b \\
b & a
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 0 \\
2 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
2 & 0 & 0 \\
1 & 2 & 1 \\
-1 & 0 & 1
\end{pmatrix}
\]
Poll

Which are diagonalizable?

\[
\begin{pmatrix}
1 & 2 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
0 & 2
\end{pmatrix}
\begin{pmatrix}
2 & 0 \\
0 & 2
\end{pmatrix}
\]
Distinct Eigenvalues

Fact. If $A$ has $n$ distinct eigenvalues, then $A$ is diagonalizable.

Why?
Non-Distinct Eigenvalues

Theorem. Suppose

- \( A = n \times n \), has eigenvalues \( \lambda_1, \ldots, \lambda_k \)
- \( a_i \) = algebraic multiplicity of \( \lambda_i \)
- \( d_i \) = dimension of \( \lambda_i \) eigenspace ("geometric multiplicity")

Then

1. \( d_i \leq a_i \) for all \( i \)
2. \( A \) is diagonalizable \( \iff \) \( \sum d_i = n \)
   \( \iff \sum a_i = n \) and \( d_i = a_i \) for all \( i \)

So: if you find one eigenvalue where the geometric multiplicity is less than the algebraic multiplicity, the matrix is not diagonalizable.
Summary of Section 6.4

- $A$ is diagonalizable if $A = CDC^{-1}$ where $D$ is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- $A$ is diagonalizable $\iff$ $A$ has $n$ linearly independent eigenvectors $\iff$ the sum of the geometric dimensions of the eigenspaces in $n$
- If $A$ has $n$ distinct eigenvalues it is diagonalizable