Announcements: November 25

- Final Exam Dec 11 6-8:50p (cumulative!)
- WeBWorK 6.6, 7.1, 7.2 due tonite
- My office hours Wed 2-3 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 250
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Tue 11-12 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 11-12 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280 Schedule
- PLUS Sessions
  - Tue/Thu 6-7 Westside Activity Room
  - Mon/Wed 6-7 GT Connector

Section 7.3

Orthogonal projection

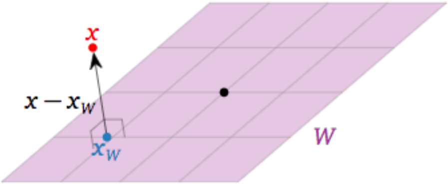
Outline of Section 7.3

- Orthogonal projections and distance
- A formula for projecting onto any subspace
- A special formula for projecting onto a line
- Matrices for projections
- Properties of projections

Orthogonal Projections

Let v be a vector in $\mathbb{R}^n$ and W a subspace of $\mathbb{R}^n$.

The orthogonal projection of v onto W the vector obtained by drawing a line segment from v to W that is perpendicular to W.



Fact. The following three things are all the same:

- The orthogonal projection of v onto W
- The vector v_W (the W-part of v)
- The closest vector in W to v

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Why? Choose $\widehat{x}$ so that $A\widehat{x} = v_W$. We know $v - v_W = v - A\widehat{x}$ is in $W^\perp = \mathrm{Nul}(A^T)$ and so

$$0 = A^T(v - A\widehat{x}) = A^T v - A^T A\widehat{x}$$

$$\rightsquigarrow A^T A\widehat{x} = A^T v$$

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

What does the theorem give when $W = \mathrm{Span}\{u\}$ is a line?

Orthogonal Projection onto a line

Special case. Let $L = \mathrm{Span}\{u\}$. For any vector v in $\mathbb{R}^n$ we have:

$$v_L = \frac{u \cdot v}{u \cdot u} u$$

Find v_L and $v_{L^\perp}$ if $v = \begin{pmatrix} -2 \\ -3 \\ -1 \end{pmatrix}$ and $u = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$.

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T A x = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Example. Find v_W if $v = \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix}$, $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Steps. Find $A^T A$ and $A^T v$, then solve for x, then compute Ax.

Question. How far is v from W?

Orthogonal Projections

Theorem. Let $W = \mathrm{Col}(A)$. For any vector v in $\mathbb{R}^n$, the equation

$$A^T Ax = A^T v$$

is consistent and the orthogonal projection v_W is equal to Ax where x is any solution.

Special case. If the columns of A are independent then $A^T A$ is invertible, and so

$$v_W = A(A^T A)^{-1} A^T v.$$

Why? The x we find tells us which linear combination of the columns of A gives us v_W. If the columns of A are independent, there's only one linear combination.

Projections as linear transformations

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(v) = v_W$ (orthogonal projection). Then

- T is a linear transformation
- $T(v) = v$ if and only if v is in W
- $T(v) = 0$ if and only if v is in $W^\perp$
- $T \circ T = T$
- The range of T is W

Matrices for projections

Fact. If the columns of A are independent and $W = \mathrm{Col}(A)$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is orthogonal projection onto W then the standard matrix for T is:

$$A(A^T A)^{-1} A^T.$$

Example. Find the standard matrix for orthogonal projection of $\mathbb{R}^3$ onto $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Properties of projection matrices

Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(v) = v_W$ (orthogonal projection). Let P be the standard matrix for T. Then

- The 1–eigenspace of P is W (unless $W = 0$)
- The 0–eigenspace of P is $W^\perp$ (unless $W = \mathbb{R}^n$)
- $P^2 = A$
- $\mathrm{Col}(P) = W$
- $\mathrm{Nul}(P) = W^\perp$
- A is diagonalizable; its diagonal matrix has m 1's & $n - m$ 0's where $m = \dim W$

You can check these properties for the matrix in the last example. It would be very hard to prove these facts without any theory. But they are all easy once you know about linear transformations!

Summary of Section 7.3

- The orthogonal projection of v onto W is v_W
- v_W is the closest point in W to v.
- The distance from v to W is $\|v_{W^\perp}\|$.
- Theorem. Let $W = \mathrm{Col}(A)$. For any v, the equation $A^T Ax = A^T v$ is consistent and v_W is equal to Ax where x is any solution.
- Special case. If $L = \mathrm{Span}\{u\}$ then $v_L = \frac{u \cdot v}{u \cdot u} u$
- Special case. If the columns of A are independent then $A^T A$ is invertible, and so $v_W = A(A^T A)^{-1} A^T v$
- When the columns of A are independent, the standard matrix for orthogonal projection to $\mathrm{Col}(A)$ is $A(A^T A)^{-1} A^T$
- Let W be a subspace of $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the function given by $T(v) = v_W$. Then
  - T is a linear transformation
  - etc.
- If P is the standard matrix then
  - The 1–eigenspace of P is W (unless $W = 0$)
  - etc.