# Announcements: October 15

- Midterm 2 Friday in recitation, §3.5–4.4
- WeBWorK due Wednesday
- Supplemental problems and Practice exam on web site
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- Review sessions
  - Talha Wed 6-7:30 Skiles 257
  - Talha Thu 6-7:30 Skiles 257
  - Arjun Thu 8-9:30 Skiles 256
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 248
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Fri 12-1 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 9:30-10:30 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- PLUS Sessions
  - Tue/Thu 6-7 Clough 280
  - Mon/Wed 7-8 Clough 123

# Review for Midterm 2

## Summary of Section 3.5

- A set of vectors $\{v_1, \ldots, v_k\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

  has only the trivial solution. It is linearly dependent otherwise.
- The cols of $A$ are linearly independent
  $\Leftrightarrow$ $Ax = 0$ has only the trivial solution.
  $\Leftrightarrow$ $A$ has a pivot in each column
- The number of pivots of $A$ equals the dimension of the span of the columns of $A$
- The set $\{v_1, v_2, \ldots, v_k\}$ is linearly dependent if and only if some $v_i$ lies in the span of $v_1, \ldots, v_{i-1}$.

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I have an $4 \times 3$ matrix $A$ and a vector $b$ in $\mathbb{R}^3$, and $Ax = b$ is inconsistent. Are the columns of $A$ linearly dependent?

# Section 3.6 Summary

- A subspace of $\mathbb{R}^n$ is a subset $V$ with:
  1. The zero vector is in $V$.
  2. If $u$ and $v$ are in $V$, then $u + v$ is also in $V$.
  3. If $u$ is in $V$ and $c$ is in $\mathbb{R}$, then $cu \in V$.
- Two important subspaces: $\mathrm{Nul}(A)$ and $\mathrm{Col}(A)$
- Find a spanning set for $\mathrm{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a spanning set for $\mathrm{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)
- Four things are the same: subspaces, spans, planes through 0, null spaces

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Let $V$ be the subset of $\mathbb{R}^3$ consisting of the $x$-axis, the $y$-axis, and the $z$-axis. Which properties of a subspace does $V$ fail?

Find a spanning set for the plane in $\mathbb{R}^3$ defined by $x + y - 2z = 0$.

# Section 3.7 Summary

- A basis for a subspace $V$ is a set of vectors $\{v_1, v_2, \ldots, v_k\}$ such that
  1. $V = \text{Span}\{v_1, \ldots, v_k\}$
  2. $v_1, \ldots, v_k$ are linearly independent
- The number of vectors in a basis for a subspace is the dimension.
- Find a basis for $\text{Nul}(A)$ by solving $Ax = 0$ in vector parametric form
- Find a basis for $\text{Col}(A)$ by taking pivot columns of $A$ (not reduced $A$)
- Basis Theorem. Suppose $V$ is a $k$-dimensional subspace of $\mathbb{R}^n$. Then
  - Any $k$ linearly independent vectors in $V$ form a basis for $V$.
  - Any $k$ vectors in $V$ that span $V$ form a basis.

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Find a basis $\{u, v, w\}$ for $\mathbb{R}^3$ where no vector has a zero entry.

# Section 3.9 Summary

- Rank-Nullity Theorem. $\operatorname{rank}(A) + \dim \operatorname{Nul}(A) = \#\operatorname{cols}(A)$

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Let $A$ be an $4 \times 6$ nonzero matrix and suppose the columns of $A$ are all the same. What is $\dim \operatorname{Nul}(A)$?

# Section 4.1 Summary

- If $A$ is an $m \times n$ matrix, then the associated matrix transformation $T$ is given by $T(v) = Av$. This is a function with domain $\mathbb{R}^n$ and codomain $\mathbb{R}^m$ and range $\mathrm{Col}(A)$.
- If $A$ is $n \times n$ then $T$ does something to $\mathbb{R}^n$; basic examples: reflection, projection, scaling, shear, rotation

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Find a matrix $A$ so that the range of the matrix transformation $T(v) = Av$ is the line $y = 2x$ in $\mathbb{R}^2$.

# Summary of Section 4.2

- $T : \mathbb{R}^n \to \mathbb{R}^m$ is one-to-one if each $b$ in $\mathbb{R}^m$ is the output for at most one $v$ in $\mathbb{R}^n$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is one-to-one
  - the columns of $A$ are
  - $Ax = 0$ has
  - $A$ has a pivot
  - the range has dimension $n$
- $T : \mathbb{R}^n \to \mathbb{R}^m$ is onto if the range of $T$ equals the codomain $\mathbb{R}^m$, that is, each $b$ in $\mathbb{R}^m$ is the output for at least one input $v$ in $\mathbb{R}^m$.
- **Theorem.** Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation with matrix $A$. Then the following are all equivalent:
  - $T$ is onto
  - the columns of $A$
  - $A$ has a pivot
  - $Ax = b$ is consistent
  - the range of $T$ has dimension $m$

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Let $A$ be an $5 \times 5$ matrix. Suppose that $\dim \operatorname{Nul}(A) = 0$. Must it be true that $Ax = e_1$ is consistent?

## Summary of 4.3

- A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear if
  - $T(u+v) = T(u) + T(v)$ for all $u, v$ in $\mathbb{R}^n$.
  - $T(cv) = cT(v)$ for all $v \in \mathbb{R}^n$ and $c$ in $\mathbb{R}$.
- **Theorem.** Every linear transformation is a matrix transformation (and vice versa).
- The standard matrix for a linear transformation has its $i$th column equal to $T(e_i)$.

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Find the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects over the line $y = -x$.

## Summary of Section 4.4

- Composition: $(T \circ U)(v) = T(U(v))$ (do $U$ then $T$)
- Matrix multiplication: $(AB)_{ij} = r_i \cdot b_j$
- Matrix multiplication: the $i$th column of $AB$ is $A(b_i)$
- The standard matrix for a composition of linear transformations is the product of the standard matrices.
- Warning!
  - $AB$ is not always equal to $BA$
  - $AB = AC$ does not mean that $B = C$
  - $AB = 0$ does not mean that $A$ or $B$ is 0

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Find a $2 \times 2$ matrix $A$, not equal to I or 0, with $A^4 = I$.

# Important terms

- linearly independent
- subspace
- column space
- null space
- basis
- dimension
- matrix transformation
- one-to-one
- onto
- linear transformation
- composition

Good luck!