Announcements: November 14

- Midterm 3 on §4.5-6.5 Friday in recitation
- WeBWorK 6.4, 6.5 due Wednesday
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 250
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Tue 11-12 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 11-12 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280 Schedule
- Review sessions
  - Talha Wed 6-7:30, Skiles 368
  - Talha Thu 6-7:30, Skiles 269
- PLUS Sessions
  - Tue/Thu 6-7 Westside Activity Room
  - Mon/Wed 6-7 Westside Activity Room

Review for Midterm 3

Summary of Section 4.5

- A is invertible if there is a matrix B (the inverse) with $AB = BA = I_n$
- If $ad - bc \neq 0$ then $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$
- If A is invertible, then $Ax = b$ has exactly one solution: $x = A^{-1}b$.
- $(A^{-1})^{-1} = A$ and $(AB)^{-1} = B^{-1}A^{-1}$
- Recipe for finding inverse: row reduce $(A \mid I_n)$.
- Say $A = n \times n$ matrix and $T : \mathbb{R}^n \to \mathbb{R}^n$ is the associated linear transformation. The following are equivalent.
  1. A is invertible
  2. T is invertible
  3. The reduced row echelon form of A is I_n, etc.

Suppose that A is a square matrix and $\det(A) = 0$. Which can you conclude?

(a) The linear transformation $T(v) = Av$ is not onto.

(b) A^2 is invertible.

(c) A cannot be row reduced.

(d) Two columns of A are equal.

(e) The column space of A is a line.

Summary of Section 5.2

- There is a recursive formula for the determinant of a square matrix:

$$\det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))$$

- We can use the same formula along any row/column.
- There are special formulas for the 2×2 and 3×3 cases.

Summary of Section 5.1

Say det is a function $\det : \{\text{matrices}\} \to \mathbb{R}$ with:

1. $\det(I_n) = 1$
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by -1
4. If we scale a row of a matrix by k, the determinant scales by k

Fact 1. There is such a function det and it is unique.

Fact 2. A is invertible $\Leftrightarrow \det(A) \neq 0$ important!

Fact 3. $\det A = (-1)^{\#\text{row swaps used}} \left(\frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right)$

Fact 4. The function can be computed by any of the $2n$ cofactor expansions.

Fact 5. $\det(AB) = \det(A)\det(B)$ important!

Fact 6. $\det(A^T) = \det(A)$

Compute the determinant.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 5 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Summary of Section 5.3

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of A.

Fact 8. If S is some subset of $\mathbb{R}^n$, then $\mathrm{vol}(T(S)) = |\det(A)| \cdot \mathrm{vol}(S)$.

Let P be the parallelogram with vertices $(1,1)$, $(2,1)$, $(3,3)$, and $(2,3)$. Let A be the matrix

$$\begin{pmatrix} 5 & 8 \\ 2 & 3 \end{pmatrix}$$

Let $T(v) = Av$ be the associated linear transformation. What is the area of $T(P)$?

Summary of Section 6.1

- If $v \neq 0$ and $Av = \lambda v$ then λ is an eigenvector of A with eigenvalue λ
- Given a matrix A and a vector v, we can check if v is an eigenvector for A: just multiply
- Given a matrix A and a number λ we can check if λ is an eigenvalue and find its eigenspace: solve $(A - \lambda I)x = 0$
- Fact. A invertible $\Leftrightarrow 0$ is not an eigenvalue of A
- Fact. If $v_1 \dots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots \lambda_k$, then $\{v_1, \dots, v_k\}$ are linearly independent.
- We can often see eigenvectors and eigenvalues without doing calculations

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Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by reflection in the line $y = 5x$. Find the eigenvalues and corresponding eigenvectors for the standard matrix of T.

Find a 3×3 matrix where e_1 is a 1-eigenvector, e_2 is a 2-eigenvector, and $e_1 + e_2$ is a 3-eigenvector.

Find a 3×3 matrix with no zero entries and with determinant 0.

Summary of Section 6.2

- The characteristic polynomial of A is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for A are the eigenvalues
- Techniques for 3×3 matrices:
  - Don't multiply out if there is a common factor
  - If there is no constant term then factor out λ
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix A:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each λ_i-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Find the eigenvalues for this bad boy:

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Summary of Section 6.4

- A is diagonalizable if $A = CDC^{-1}$ where D is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- A is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $\Leftrightarrow$ the sum of the geometric dimensions of the eigenspaces in n
- If A has n distinct eigenvalues it is diagonalizable

Answer Yes/No/Maybe:

Suppose that A is a 4×4 matrix with eigenvalues 0, 1, 2, and 3. Is A diagonalizable?

Suppose that A is a 4×4 matrix with eigenvalues 0, 1, 2 and the dimension of the 2-eigenspace is 2. Is A diagonalizable?

Suppose that A is a 4×4 matrix with eigenvalues 0, 1, i and $-i$. Is A diagonalizable?

Suppose that A is a 4×4 matrix where the algebraic multiplicity of the eigenvalue 0 is 2 and the null space of A is a line. Is A diagonalizable?

Summary of Section 6.5

- Complex numbers allow us to solve all polynomials completely, and find n eigenvectors for an $n \times n$ matrix
- If λ is an eigenvalue with eigenvector v then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$

Say that A is a matrix with i-eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$.

Is $\begin{pmatrix} i \\ 1 \end{pmatrix}$ an eigenvector for the matrix?

Good luck!