Announcements: November 14

- Midterm 3 on §4.5-6.5 Friday in recitation
- WeBWorK 6.4, 6.5 due Wednesday
- My office hours Wed 2-3 and Friday 9:30-10:30 in Skiles 234
- TA office hours
  - Arjun Wed 3-4 Skiles 230
  - Talha Tue/Thu 11-12 Clough 250
  - Athreya Tue 3-4 Skiles 230
  - Olivia Thu 3-4 Skiles 230
  - James Tue 11-12 Skiles 230
  - Jesse Wed 9:30-10:30 Skiles 230
  - Vajraang Thu 11-12 Skiles 230
  - Hamed Thu 11:15-12, 1:45-2:45, 3-4:15 Clough 280
- Math Lab Monday-Thursday 11:15-5:15 Clough 280
- Review sessions
  - Talha Wed 6-7:30, Skiles 368
  - Talha Thu 6-7:30, Skiles 269
- PLUS Sessions
  - Tue/Thu 6-7 Westside Activity Room
  - Mon/Wed 6-7 Westside Activity Room
Review for Midterm 3
Summary of Section 4.5

- \( A \) is invertible if there is a matrix \( B \) (the inverse) with \( AB = BA = I_n \)
- If \( ad - bc \neq 0 \) then \( \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \)
- If \( A \) is invertible, then \( Ax = b \) has exactly one solution: \( x = A^{-1}b \).
- \((A^{-1})^{-1} = A \) and \((AB)^{-1} = B^{-1}A^{-1}\)
- Recipe for finding inverse: row reduce \((A | I_n)\).
- Say \( A = n \times n \) matrix and \( T : \mathbb{R}^n \to \mathbb{R}^n \) is the associated linear transformation. The following are equivalent.
  1. \( A \) is invertible
  2. \( T \) is invertible
  3. The reduced row echelon form of \( A \) is \( I_n \), etc.

Suppose that \( A \) is a square matrix and \( \det(A) = 0 \). Which can you conclude?

(a) The linear transformation \( T(v) = Av \) is not onto.
(b) \( A^2 \) is invertible.
(c) \( A \) cannot be row reduced.
(d) Two columns of \( A \) are equal.
(e) The column space of \( A \) is a line.
Summary of Section 5.2

- There is a recursive formula for the determinant of a square matrix:
  \[
  \det(A) = a_{11}(\det(A_{11})) - a_{12}(\det(A_{12})) + \cdots \pm a_{1n}(\det(A_{1n}))
  \]
- We can use the same formula along any row/column.
- There are special formulas for the 2 × 2 and 3 × 3 cases.
Summary of Section 5.1

Say \( \text{det} \) is a function \( \text{det} : \{ \text{matrices} \} \to \mathbb{R} \) with:

1. \( \text{det}(I_n) = 1 \)
2. If we do a row replacement on a matrix, the determinant is unchanged
3. If we swap two rows of a matrix, the determinant scales by \(-1\)
4. If we scale a row of a matrix by \(k\), the determinant scales by \(k\)

Fact 1. There is such a function \( \text{det} \) and it is unique.

Fact 2. \( A \) is invertible \(\iff\) \( \text{det}(A) \neq 0 \) important!

Fact 3. \( \text{det} A = (-1)^{\# \text{row swaps used}} \left( \frac{\text{product of diagonal entries of row reduced matrix}}{\text{product of scalings used}} \right) \)

Fact 4. The function can be computed by any of the \(2n\) cofactor expansions.

Fact 5. \( \text{det}(AB) = \text{det}(A) \text{det}(B) \) important!

Fact 6. \( \text{det}(A^T) = \text{det}(A) \)

Compute the determinant.

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 5 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]
Summary of Section 5.3

Fact 7. $\det(A)$ is signed volume of the parallelepiped spanned by cols of $A$.

Fact 8. If $S$ is some subset of $\mathbb{R}^n$, then $\text{vol}(T(S)) = |\det(A)| \cdot \text{vol}(S)$.

Let $P$ be the parallelogram with vertices $(1, 1), (2, 1), (3, 3)$, and $(2, 3)$. Let $A$ be the matrix

$$
\begin{pmatrix}
5 & 8 \\
2 & 3
\end{pmatrix}
$$

Let $T(v) = Av$ be the associated linear transformation. What is the area of $T(P)$?
Summary of Section 6.1

• If $v \neq 0$ and $Av = \lambda v$ then $\lambda$ is an eigenvector of $A$ with eigenvalue $\lambda$

• Given a matrix $A$ and a vector $v$, we can check if $v$ is an eigenvector for $A$: just multiply

• Given a matrix $A$ and a number $\lambda$ we can check if $\lambda$ is an eigenvalue and find its eigenspace: solve $(A - \lambda I)x = 0$

• Fact. $A$ invertible $\iff 0$ is not an eigenvalue of $A$

• Fact. If $v_1 \ldots v_k$ are distinct eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then $\{v_1, \ldots, v_k\}$ are linearly independent.

• We can often see eigenvectors and eigenvalues without doing calculations

Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the linear transformation given by reflection in the line $y = 5x$. Find the eigenvalues and corresponding eigenvectors for the standard matrix of $T$.

Find a $3 \times 3$ matrix where $e_1$ is a 1-eigenvector, $e_2$ is a 2-eigenvector, and $e_1 + e_2$ is a 3-eigenvector.

Find a $3 \times 3$ matrix with no zero entries and with determinant 0.
Summary of Section 6.2

- The characteristic polynomial of $A$ is $\det(A - \lambda I)$
- The roots of the characteristic polynomial for $A$ are the eigenvalues
- Techniques for $3 \times 3$ matrices:
  - Don’t multiply out if there is a common factor
  - If there is no constant term then factor out $\lambda$
  - If the matrix is triangular, the eigenvalues are the diagonal entries
  - Guess one eigenvalue using the rational root theorem, reverse engineer the rest (or use long division)
  - Use the geometry to determine an eigenvalue
- Given an square matrix $A$:
  - The eigenvalues are the solutions to $\det(A - \lambda I) = 0$
  - Each $\lambda_i$-eigenspace is the solution to $(A - \lambda_i I)x = 0$

Find the eigenvalues for this bad boy:

$$
\begin{pmatrix}
0 & -1 & -1 \\
1 & 2 & 1 \\
1 & 1 & 2
\end{pmatrix}
$$
Summary of Section 6.4

- $A$ is diagonalizable if $A = CDC^{-1}$ where $D$ is diagonal
- A diagonal matrix stretches along its eigenvectors by the eigenvalues, similar to a diagonal matrix
- If $A = CDC^{-1}$ then $A^k = CD^kC^{-1}$
- $A$ is diagonalizable $\iff$ $A$ has $n$ linearly independent eigenvectors $\iff$ the sum of the geometric dimensions of the eigenspaces in $n$
- If $A$ has $n$ distinct eigenvalues it is diagonalizable

Answer Yes/No/Maybe:

Suppose that $A$ is a $4 \times 4$ matrix with eigenvalues 0, 1, 2, and 3. Is $A$ diagonalizable?

Suppose that $A$ is a $4 \times 4$ matrix with eigenvalues 0, 1, 2 and the dimension of the 2-eigenspace is 2. Is $A$ diagonalizable?

Suppose that $A$ is a $4 \times 4$ matrix with eigenvalues 0, 1, $i$ and $-i$. Is $A$ diagonalizable?

Suppose that $A$ is a $4 \times 4$ matrix where the algebraic multiplicity of the eigenvalue 0 is 2 and the null space of $A$ is a line. Is $A$ diagonalizable?
Summary of Section 6.5

- Complex numbers allow us to solve all polynomials completely, and find \( n \) eigenvectors for an \( n \times n \) matrix
- If \( \lambda \) is an eigenvalue with eigenvector \( v \) then \( \bar{\lambda} \) is an eigenvalue with eigenvector \( \bar{v} \)

Say that \( A \) is a matrix with \( i \)-eigenvector \( \begin{pmatrix} 1 \\ i \end{pmatrix} \).

Is \( \begin{pmatrix} i \\ 1 \end{pmatrix} \) an eigenvector for the matrix?
Good luck!