Math 1553 Worksheet: Fundamentals and §2.1

Solutions

1. For each equation, determine whether the equation is linear or non-linear. Circle your answer. If the equation is non-linear, briefly justify why it is non-linear.

   a) $3x_1 + \sqrt{x_2} = 4$ Linear   Not linear

   b) $x_1 = x_2 - x_3 + 10x_4$ Linear   Not linear

   c) $e^{\pi}x + \ln(13)y = \sqrt{2} - z$ Linear   Not linear

Solution.

   a) Not linear. The $\sqrt{x_2}$ term makes it non-linear.

   b) Linear.

   c) Linear. Note e^{π} and $\sqrt{2}$ are just real numbers. Don't be misled by the appearance of the natural logarithm: $\ln(13)$ is just the coefficient for y.
      If the second term had been $\ln(13y)$ instead of $\ln(13)y$, then y would have been inside the logarithm and the equation would have been non-linear.

2. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$
\begin{aligned}
2x + 4y + 4z &= 1 \\
2x + 5y + 2z &= -1 \\
y + 3z &= 8
\end{aligned}
$$

Do all three of the planes intersect? If so, do they intersect at a single point, a line, or a plane?

Solution.

Subtracting the first equation from the second gives us

$$
\begin{aligned}
2x + 4y + 4z &= 1 \\
y - 2z &= -2 \\
y + 3z &= 8.
\end{aligned}
$$

Next, subtracting the second equation from the third gives us

$$
\begin{aligned}
2x + 4y + 4z &= 1 \\
y - 2z &= -2 \\
5z &= 10,
\end{aligned}
$$

so $z = 2$. We can back-substitute to find y and then x. The second equation is $y - 2z = -2$, so $y - 2(2) = -2$, thus $y = 2$. The first equation is $2x + 4(2) + 4(2) = 1$, so $2x = -15$, thus $x = -15/2$. We have found that the planes intersect at the point

$$
\left(-\frac{15}{2}, \ 2, \ 2 \right).
$$

An alternative method would have been to use augmented matrices to isolate z and then back-substitute:

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

The last line is $5z = 10$, so $z = 2$. From here, back-substitution would give us $y = 2$ and then $x = -\frac{15}{2}$, just like before.

3. Find all values of h so that the lines $x + hy = -5$ and $2x - 8y = 6$ do *not* intersect. For all such h, draw the lines $x + hy = -5$ and $2x - 8y = 6$ to verify that they do not intersect.

Solution.

We can use basic algebra, row operations, or geometric intuition.

Using basic algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$x + hy = -5$$
$$2x - 8y = 6.$$

Subtracting twice the first equation from the second equation gives us

$$x + hy = -5$$
$$(-8 - 2h)y = 16.$$

If $-8 - 2h = 0$ (so $h = -4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h = -4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8 - 2h)y = 16 \qquad y = \frac{16}{-8 - 2h} \qquad y = \frac{8}{-4 - h}.$$

We can now substitute this value of y into the first equation to find x at the point of intersection:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4 - h} = -5 \qquad x = -5 - \frac{8h}{-4 - h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h = -4}$.

Using intuition from geometry in $\mathbf{R}^2$: Two non-identical lines in $\mathbf{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$. If $h \neq 0$, then the first line is $y = -\frac{1}{h}x - \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h = -4$. In this case, the lines are $y = \frac{1}{4}x + \frac{5}{4}$ and $y = \frac{1}{4}x - \frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h = 0$ then the first line is vertical and the two lines intersect when $x = -5$).

Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \xrightarrow{R_2=R_2-2R_1} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$
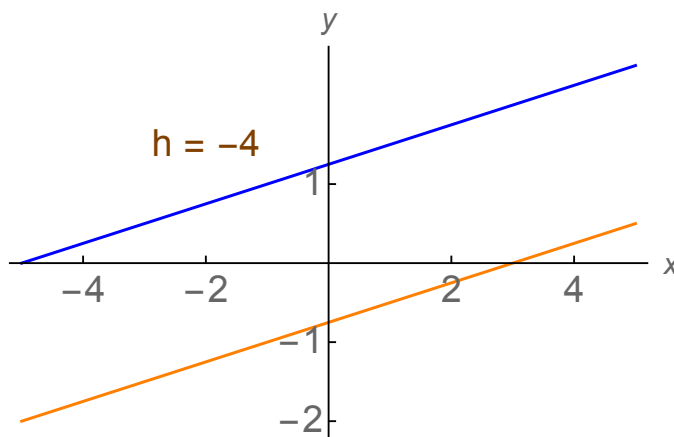
If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y=16$, so

$$y = \frac{16}{-8-2h} = \frac{8}{-4-h} \quad \text{and} \quad x = -5-hy = -5-\frac{8h}{-4-h},$$

and the lines intersect at (x, y). Therefore, our answer is $h=-4$.

Here are the two lines for $h=-4$, and we can see they are different parallel lines.

If we vary h away from -4, then the blue and orange lines will have different slopes and will inevitably intersect. For example,