Math 1553 Worksheet §4.4, Matrix Multiplication

Solutions

1. If A is a 3×5 matrix and B is a 3×2 matrix, which of the following are defined? Very briefly justify your answer.

a) $A - B$

b) AB

c) $A^T B$

d) $B^T A$

e) A^2

Solution.

Only (c) and (d).

a) $A - B$ is nonsense. In order for $A - B$ to be defined, A and B need to have the same number of rows and same number of columns as each other.

b) AB is undefined since the number of columns of A does not equal the number of rows of B.

c) A^T is 5×3 and B is 3×2, so $A^T B$ is a 5×2 matrix.

d) B^T is 2×3 and A is 3×5, so $B^T A$ is a 2×5 matrix.

e) A^2 is nonsense (can't do 3×5 times 3×5).

2. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) Suppose A and B are matrices and the matrix product AB is defined. Then each column of AB must be a linear combination of the columns of A.

b) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^2$ and codomain $\mathbf{R}^3$.

c) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ and $U : \mathbf{R}^m \to \mathbf{R}^p$ are linear transformations and $U \circ T$ is onto. Then U and T must both be onto.

Solution.

a) True. If we let $v_1, \ldots, v_p$ be the columns of B, then $AB = \begin{pmatrix} Av_1 & Av_2 & \cdots & Av_p \end{pmatrix}$, where Av_i is in the column span of A for every i (this is part of the definition of matrix multiplication of vectors).

b) True. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

c) False. Take the linear transformations $T : \mathbf{R}^3 \to \mathbf{R}^3$ and $U : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x, y, 0)$ and $U(x, y, z) = (x, y)$. Then $(U \circ T)(x, y, z) = (x, y)$, so

$U \circ T$ maps $\mathbf{R}^3$ onto $\mathbf{R}^2$. However, T is not onto since the z-coordinate of every vector in its image is 0.

3. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be rotation *clockwise* by $60°$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation with standard matrix $\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}$.

a) Find the standard matrix for the composition $U \circ T$.

b) Find the standard matrix for the composition $T \circ U$.

c) Is rotating clockwise by $60°$ and then performing U, the same as first performing U and then rotating clockwise by $60°$?

Solution.

a) The matrix for T is $\begin{pmatrix} \cos(-60°) & -\sin(-60°) \\ \sin(-60°) & \cos(-60°) \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$. The matrix for $U \circ T$ is

$$\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 - \frac{\sqrt{3}}{2} & \frac{1}{2} - \sqrt{3} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}.$$

b) The matrix for $T \circ U$ is

$$\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 + \frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

c) No. In (a) and (b), we found that the standard matrices for $U \circ T$ and $T \circ U$ are different, so the transformations are different.