Math 1553 Worksheet: 4.5, 5.1-5.3

1. Answer true if the statement is *always* true. Otherwise, answer false.

a) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

b) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

c) Suppose A and B are invertible $n \times n$ matrices. Then $A+B$ is invertible and

$$(A+B)^{-1} = A^{-1} + B^{-1}.$$

Solution.

a) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, this is the same as saying A is invertible, so T is one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

b) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

c) False. $A+B$ might not be invertible in the first place. For example, if $A = I_2$ and $B = -I_2$ then $A+B = 0$ which is not invertible. Even in the case when $A+B$ is invertible, it still might not be true that $(A+B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

2. Find the volume of the parallelepiped naturally formed by $\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$.

Solution.

We compute

$$\begin{aligned} \det\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 3 \\ -2 & 1 & 1 \end{pmatrix} &= 2\det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix} - 1\det\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} + 1\det\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \\ &= 2(2-3) - 1(1+6) + 1(1+4) \\ &= -2 - 7 + 5 = -4. \end{aligned}$$

The volume is $|-4| = 4$.

3. Let $A = \begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix}$.

a) Compute $\det(A)$ using row reduction.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

Solution.

a) Below, r counts the row swaps and s measures the scaling factors.

$$\begin{pmatrix} 2 & -8 & 6 & 8 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \xrightarrow{R_1 = \frac{R_1}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 3 & -9 & 5 & 10 \\ -3 & 0 & 1 & -2 \\ 1 & -4 & 0 & 6 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow[R_3 = R_3 + 3R_1,\ R_4 = R_4 - R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & -12 & 10 & 10 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_3 = R_3 + 4R_2} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & -3 & 2 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\xrightarrow{R_4 = R_4 - \frac{R_2}{2}} \begin{pmatrix} 1 & -4 & 3 & 4 \\ 0 & 3 & -4 & -2 \\ 0 & 0 & -6 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (r = 0,\ s = \frac{1}{2})$$

$$\det(A) = (-1)^0 \frac{1 \cdot 3 \cdot (-6) \cdot 1}{1/2} = -36.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{36}$.

c) $\det(A^T) = \det(A) = -36$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-36)^5 = -60,466,176$$

4. Play matrix tic-tac-toe!

Instead of X against O, we have 1 against 0. The 1-player wins if the final matrix has nonzero determinant, while the 0-player wins if the determinant is zero. You can change who goes first, and you can also modify the size of the matrix.

Click the link above, or copy and paste the url below:

`http://textbooks.math.gatech.edu/ila/demos/tictactoe/tictactoe.html`

Can you think of a winning strategy for the 0 player who goes first in the 2×2 case? Is there a winning strategy for the 1 player if they go first in the 2×2 case?