② $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ <u>neither</u>   not all the zero rows are at the bottom

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ <u>REF</u>   not all the pivots are 1's

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ <u>neither</u>   does not follow the staircase pattern ⌐⌐

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ <u>RREF</u>

③ $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 2 & 0 \\ 1 & 3 & h \end{bmatrix}$ $\xrightarrow{R_2 \to \frac{1}{2}R_2}$ $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 1 & 3 & h \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - R_1}$

$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 2 & h-3 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - 2R_2}$ $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & h-3 \end{bmatrix}$

$h = 3$ in order to make X_3 a free variable. With one free variable, the solution set will be a line.

④ $\begin{bmatrix} 1 & 3 & 0 & | & -2 \\ 0 & 0 & 1 & | & 4 \end{bmatrix}$

$x_1 + 3x_2 + 0x_3 = -2$

$0x_1 + 0x_2 + x_3 = 4$

$x_1 = -2 - 3x_2$

$x_2 = x_2$ (free /real)

$x_3 = 4$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2-3x_2 \\ x_2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}$

$A = -2 \qquad D = -3$

$B = 0 \qquad E = 1$

$C = 4 \qquad F = 0$

⑤ A) True

The solution set being a line means there is one free variable. This does not change between $Ax = b$ and $Ax = 0$. The line is just translated to pass through 0 instead of b.

B) False

counterexample:

$Ax = b$ inconsistent

$\begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 0 & | & 1 \end{bmatrix}$

$Ax = 0$ consistent

$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$ $\quad x_1 = 0$

$x_2 = x_2$ (free)

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & h \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & h \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 1 & h \end{bmatrix} \xrightarrow{R_3 \to R_3 - 2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & h-2 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & h-2 \end{bmatrix}$$

Not linear Independent   If there is not a
pivot   in   every   column.   Thus,   not
linear   Independent   If   h = 2.

⑦ 5 × 7 matrix $\equiv \begin{bmatrix} & \ \ \ \ \ \ \ \\ & \end{bmatrix}$

A) False
   The   maximum   number   of   pivots   would be 5.
   There   are   7   columns.
   7 > 5,   so   there   cannot   be a   pivot in
   every   column.

B) True
   The   minimum   number   of   free   variables
   would   be   2   because   7 − 5 = 2.
   Thus,   you   would   need   two   different
   vectors   for   each   free   variable   in   the
   solution   set.

C) True.   we   can   provide   an   example
   $$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$
   Thus,   there   is   no   potential   for   a   row
   of   zeros   in   A.

(8)

$$\begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 3 & -2 & 1 & | & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 0 & -5 & 1 & | & -2 \end{bmatrix} \xrightarrow{R_2 \to -\frac{1}{5} R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 1 \\ 0 & 1 & -1/5 & | & 2/5 \end{bmatrix} \to R_1 \to R_1 - R_2$$

$$\begin{bmatrix} 1 & 0 & 1/5 & | & 3/5 \\ 0 & 1 & -1/5 & | & 2/5 \end{bmatrix}$$

2 pivots and 1 free variable

Thus, the solution set is a line.
Thus, there are infinitely many solutions.

Note: Because there are two pivots,
span $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ would be a plane.
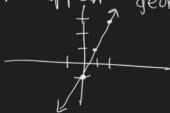
(9) False

This is only true when the solution set contains the origin. You could have the following solution set counterexample:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This would appear geometrically as:

which does not contain the origin & cannot be a span.

⑩ A) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 5 & 6 & 9 \end{bmatrix}$ $\xrightarrow{R_2 \to R_2 - 2R_1}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -3 \\ 3 & 5 & 6 & 9 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 3R_1}$ $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -3 \\ 0 & -1 & -3 & -3 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - R_2}$

$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ TWO pivots, so Span $\{col (A)\}$ is a plane in $\mathbb{R}^3$ __NO__

B) $\begin{bmatrix} 1 & 0 & 0 \\ 18 & 2 & 0 \\ 20 & 1 & 12 \end{bmatrix}$ $\xrightarrow{R_2 \to R_2 - 18R_1}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 20 & 1 & 12 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - 20R_1}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 12 \end{bmatrix}$ $\xrightarrow{R_2 \to \frac{1}{2}R_2}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 12 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - R_2}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 12 \end{bmatrix}$ 3 pivots and 3 rows __yes__

C) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\xrightarrow{R_2 \to \frac{1}{2}R_2}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\xrightarrow{R_3 \to R_3 - R_2}$

$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 3 pivots and 3 rows __yes__

D) This does reduce to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

but it does not span $\mathbb{R}^3$ because there are 4 rows and thus the span is in $\mathbb{R}^4$. The span is 3-dimensional, but it is not described as spanning $\mathbb{R}^3$. __NO__

(11) 3 Equations and 3 variables $= \begin{bmatrix} 1 & 1 & 1 \\ & & \end{bmatrix}$

point
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ 0 free variables **yes**

plane
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 2 free variables **yes**

line
$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ 1 free variable **yes**

NO
solution $[A \mid b] = \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}$ **yes**

Question 12. We can plug in each of the points $(1, 2)$, $(2, 1)$, and $(3, 0)$ in the equations to see if the system is satisfied.

$$(1, 2): \begin{cases} 1 + 2 & = 3 \\ 2 & \neq 1 \end{cases}, \qquad (2, 1): \begin{cases} 2 + 1 & = 3 \\ 1 & = 1 \end{cases}, \qquad (3, 0): \begin{cases} 3 + 0 & = 3 \\ 0 & \neq 1 \end{cases},$$

Out of the three options, only the point $(2, 1)$ is a solution of the system.

Question 13. When we represent the system in matrix form $Ax = b$, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Both matrices $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $[\, A \mid b \,] = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$ have the same number of pivots, so the system is consistent and has a solution. Moreover, there are two pivots, which means that there is one free variable. We conclude that the solution space is a line in $\mathbb{R}^3$.

Alternatively, we can consider the geometric shape of each equation of the system separately, and then the solution to the system will be the intersection of these two geometric shapes. The equation $x + y = 2$ is a plane in $\mathbb{R}^3$ parallel to the z-axis. The equation $z = 1$ is a plane in $\mathbb{R}^3$ perpendicular to the z-axis. The two planes are not parallel to each other, so they must meet at a line. So, the solution to the system is a line. The following picture is an illustration of this fact. The red line at the intersection of the two planes is the solution:
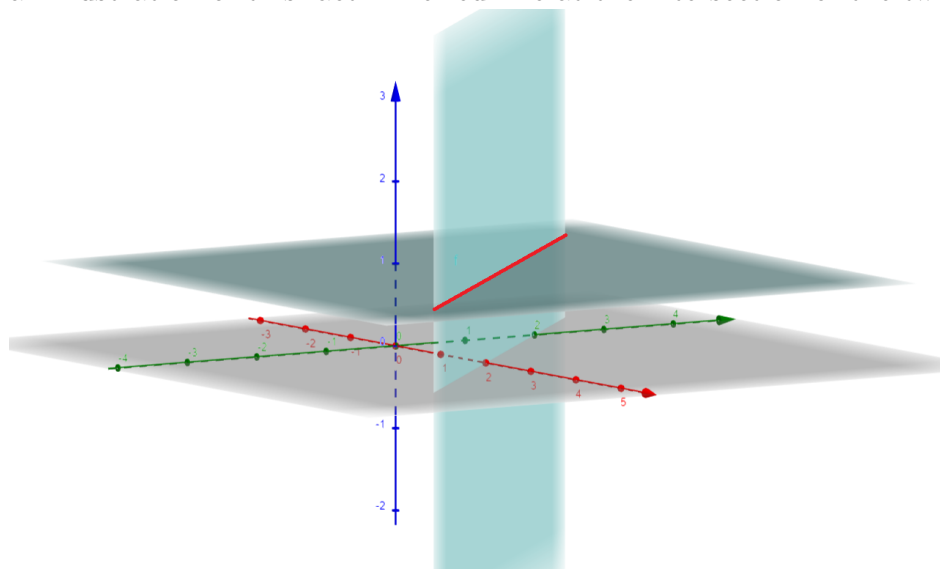
Question 14. We may plug in the given points $(0, 0)$, $(1, -1)$, and $(0, -7)$ into the equation $(x^2 + y^2) + Bx + Cy + D = 0$ to get the following equations in B, C, and D:

$$\begin{cases} (0^2 + 0^2) + 0 \cdot B + 0 \cdot C + D = 0 \\ (1^2 + (-1)^2) + 1 \cdot B + (-1) \cdot C + D = 0 \\ (0^2 + (-7)^2) + 0 \cdot B + (-7) \cdot C + D = 0 \end{cases}$$

which simplifies to

$$\begin{cases} \phantom{B - C + {}} D = 0 \\ B - C + D = -2 \\ \phantom{B - {}} -7C + D = -49 \end{cases}$$

We can solve this by forming an augmented matrix and row-reducing it as

$$\left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & -2 \\ 0 & -7 & 1 & -49 \end{array}\right]$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & -7 & 1 & -49 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & -7 & 1 & -49 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{R_1 = R_1 - R_3 \\ R_2 = R_2 - R_3}]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & -7 & 0 & -49 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 = (-1/7)R_2]{} \left[\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_1 = R_1 + R_2]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

so we get the solution $B = 5$, $C = 7$, and $D = 0$.

Question 15. We want to find a 3×2 matrix A and a vector b in $\mathbb{R}^3$ such that the augmented matrix $[\,A \mid b\,]$ is in RREF and the linear system $A \begin{bmatrix} x \\ y \end{bmatrix} = b$ has the solution $y = -3x$, which we rewrite as $3x + y = 0$.

Since there is one free variable, we see that A has exactly one pivot. Since each non-zero row of $[\,A \mid b\,]$ has a pivot, there must be two rows of $[\,A \mid b\,]$ which are the zero row. And these are the bottom two rows, because the matrix is in RREF.

Finally, the first row corresponds to the equation $3x + y = 0$, so the first row is a multiple of $\begin{bmatrix} 3 & 1 \mid 0 \end{bmatrix}$. For the row to have a leading 1, it must be multiplied $1/3$ to get $\begin{bmatrix} 1 & 1/3 \mid 0 \end{bmatrix}$. Therefore,

$$[\,A \mid b\,] = \left[\begin{array}{cc|c} 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Question 16. False. The number of vectors says nothing about linear independence. For

example, take the set of 5 vectors

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} , \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

in $\mathbb{R}^7$. Clearly, this set is linearly dependent.

Question 17. The first two options are correct; they are indeed equivalent to stating that the columns of A are linearly independent.

The third and fourth options are not equivalent, however. It is true that *if* the columns of A are linearly independent, *then* no two columns of A are equal or multiples of each other. However, the converse does not hold; just because no two columns of A are equal or multiples of each other, that does *not* imply that the columns of A are linearly independent. For example, take the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We can verify that no two columns of A are equal or multiples of each other. However, the columns of A are not linearly independent:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(We can also verify that A does not have a pivot in the third column, and that $Ax = 0$ has a non-zero solution $\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$. This is consistent with our answer that the first two options are equivalent to linear independence of the columns of A.)

Question 18. We can form augmented matrices for each system and find the number of pivots. Then the number of free variables is the total number of variables minus the number of pivots.

The first system can be written as

$$\begin{bmatrix} 1 & -1 & 8 \,\big|\, -6 \end{bmatrix}$$

which has 1 pivot (the first column), so there are $3 - 1 = 2$ free variables.

The second system becomes

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 0 \,\big|\, 7 \\ 0 & 0 & 1 & -1 & 0 \,\big|\, 0 \\ 0 & 0 & 0 & 0 & 1 \,\big|\, 5 \end{bmatrix}$$

3

which has 3 pivots in columns 1, 3, and 5, so there are $5 - 3 = 2$ free variables.

The third system,

$$\left[\begin{array}{cccccc|c} 1 & -1 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

also has 3 pivots in columns 1, 3, and 5, so there are $6 - 3 = 3$ free variables.

Question 19. Note that the free variables are x_2 and x_4. We can rewrite the augmented matrix as the equations

$$\begin{cases} x_1 - 2x_2 \quad\;\; - 5x_4 \;\; = 0 \\ \qquad\qquad x_3 \;\; + x_4 \;\; = 0 \end{cases}$$

and we can solve for x_1 and x_3 in terms of x_2 and x_4. From the first equation, we get $x_1 = 2x_2 + 5x_4$, and from the second equation, we get $x_3 = -x_4$. Therefore we have

$$\begin{cases} x_1 = 2x_2 + 5x_4 \\ x_2 = \quad x_2 \\ x_3 = \qquad\quad -x_4 \\ x_4 = \qquad\quad\; x_4 \end{cases}$$

Question 20. We recognize that the solution to $Ax = b$ is a translation of the solution to $Ax = 0$. Since the plane $z = 1$ is parallel to the xy-plane, the solution to $Ax = 0$ is also a plane parallel to the xy-plane. Moreover, the solution to $Ax = 0$ includes the zero vector, so the solution passes through the origin. We conclude that the solution to $Ax = 0$ coincides with the xy-plane.

Question 21. The three nearest nodes to the node x are the nodes 4, y, and z. The average of these three nodes equals x, so we have

$$x = \frac{4 + y + z}{3}$$

which simplifies to $3x - y - z = 4$. We can do the same for nodes y and z to get the equations

$$y = \frac{0 + z + x}{3}, \quad z = \frac{8 + x + y}{3}$$

which each simplifies to $x - 3y + z = 0$ and $x + y - 3z = -8$. So we get a system of 3 equations in 3 variables x, y, z:

$$\begin{cases} 3x - y - z = 4 \\ x - 3y + z = 0 \\ x + y - 3z = -8 \end{cases}$$

4

We can now solve for x, y, and z by row-reduction:

$$\begin{bmatrix} 3 & -1 & -1 & | & 4 \\ 1 & -3 & 1 & | & 0 \\ 1 & 1 & -3 & | & -8 \end{bmatrix}$$

$$\xrightarrow[R_1 \leftrightarrow R_2]{} \begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 3 & -1 & -1 & | & 4 \\ 1 & 1 & -3 & | & -8 \end{bmatrix}$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{} \begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 1 & 1 & -3 & | & -8 \\ 3 & -1 & -1 & | & 4 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 = R_2 - R_1 \\ R_3 = R_3 - 3R_1}]{} \begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 0 & 4 & -4 & | & -8 \\ 0 & 8 & -4 & | & 4 \end{bmatrix}$$

$$\xrightarrow[\substack{R_2 = (1/4)R_2 \\ R_3 = (1/4)R_3}]{} \begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 0 & 1 & -1 & | & -2 \\ 0 & 2 & -1 & | & 1 \end{bmatrix}$$

$$\xrightarrow[R_3 = R_3 - 2R_2]{} \begin{bmatrix} 1 & -3 & 1 & | & 0 \\ 0 & 1 & -1 & | & -2 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$$

$$\xrightarrow[\substack{R_1 = R_1 - R_3 \\ R_2 = R_2 + R_3}]{} \begin{bmatrix} 1 & -3 & 0 & | & -5 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$$

$$\xrightarrow[R_1 = R_1 + 3R_2]{} \begin{bmatrix} 1 & 0 & 0 & | & 4 \\ 0 & 1 & 0 & | & 3 \\ 0 & 0 & 1 & | & 5 \end{bmatrix}$$

so $x = 4$, $y = 3$, and $z = 5$.