Question 2. Denote the matrix by A. We recall that the characteristic polynomial is the determinant of $A - \lambda I$ as a function of λ, so we compute

$$\det(A - \lambda I) = \det \begin{pmatrix} 6-\lambda & 9 \\ -4 & -6-\lambda \end{pmatrix} = (6-\lambda)(-6-\lambda) - (-4)9 = \lambda^2 - 36 + 36 = \lambda^2$$

Question 3. We recall that the steady state vector is the eigenvector of the matrix corresponding to $\lambda = 1$ with entries that sum to 1. The 1-eigenspace of the matrix is the null space of $A - 1(I)$, which we find by doing

$$A - I = \begin{pmatrix} -1/3 & 1/3 \\ 1/3 & -1/3 \end{pmatrix} \overset{RREF}{\Longrightarrow} \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \implies x_1 = x_2$$

We see that x_1 is a pivot variable and x_2 is free. Putting this in parametric vector form gives us

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so the 1-eigenspace is span$(\begin{pmatrix} 1 \\ 1 \end{pmatrix})$. The element in this eigenspace with elements that sum to 1 is $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$, which is the steady state vector.

Question 4. We recall that, for a 2x2 matrix, we have

$$A^{-1} = \frac{1}{det(A)} C^T$$

where C is the cofactor matrix, so we can compute

$$A^{-1} = \frac{1}{1/2} \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix}$$

Then, we may find x as

$$x = A^{-1} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$$

Question 5. Since A is $2x2$ and we have found 2 linearly independent eigenvectors v_1, v_2 of A corresponding to the eigenvalues λ_1, λ_2, we can form the eigendecomposition CDC^{-1}. These matrices are

$$C = \begin{pmatrix} v_1 & v_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \; D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

so we compute

$$A = CDC^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 2 \end{pmatrix}$$

Question 6. In order for a 2x2 matrix to not be diagonalizable, it must not have 2 linearly independent eigenvalues, which requires a repeated real eigenvalue. The characteristic polynomial of this matrix is

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & k \\ 2 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) - 2k = \lambda^2 - 2\lambda - (3+2k)$$

which, by completing the square, is equal to $(\lambda - 1)^2$ if $3 + 2k = -1 \implies k = -2$. For this value of k, the characteristic polynomial has a repeated root, so A will have a repeated eigenvalue. Computing the 1-eigenspace and seeing that it is of dimension 1 will show that this is not diagonalizable.

Question 7. We first form the standard matrix A for the transformation T. Using the standard vectors e_1, e_2, we see that

$$T(e_1) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \; T(e_2) = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \implies A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

We then find the eigenvalues of T by computing the roots of the characteristic polynomial, which is

$$\det(A - \lambda I) = \det \begin{pmatrix} 3-\lambda & -2 \\ 4 & -1-\lambda \end{pmatrix} = (3-\lambda)(-1-\lambda) - (-2)4 = \lambda^2 - 2\lambda + 5$$

By using the quadratic formula, the roots are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{2 \pm \sqrt{4 - 20}}{2} = 1 \pm 2i$$

Question 8. The following row reduction steps take you from the first matrix to the second

1. Swap rows 1 and 2 (multiply determinant by -1)
2. Scale row 1 by a factor of 3 (multiply determinant by 3)
3. Scale row 2 by a factor of 2 (multiply determinant by 2)
4. Add row 2 to row 1 (no change to determinant)

The new determinant is thus $1 \cdot -1 \cdot 3 \cdot 2 = -6$.

Question 9. This triangle is the result of removing the triangle with vertices at $(1,2),(3,2),(3,5)$ from the triangle with vertices at $(1,2),(3,2),(3,6)$. These are both right triangles, so we can easily compute their area, and the area of the difference is $\frac{1}{2}(2*4-2*3)=1$.

Question 10. We can compute the determinant via cofactor expansion on the first column, which is

$$\begin{aligned}
\det(A) &= 2\det\begin{pmatrix}3 & k\\ k & -1\end{pmatrix} + (-1)\begin{pmatrix}4 & 4\\ 3 & k\end{pmatrix}\\
&= 2(-3-k^2)-(4k-12)\\
&= -2k^2-4k+6 = 8 \implies k^2+2k+1=0 \implies k=-1
\end{aligned}$$

Question 11. We recall that, for a 2x2 matrix, the negative of the trace is the coefficient of λ in the characteristic polynomial, and the determinant is the constant in the characteristic polynomial, so we must have

$$p(\lambda) = \lambda^2 - 5\lambda + 0 = \lambda(\lambda-5)$$

The eigenvalues are the roots of this polynomial, which are 0 and 5, so the smaller eigenvalue is 0 and the larger eigenvalue is 5. We can verify this by checking that their sum is the trace, 5, and their product is the determinant, 0.

Question 12. First, note that all of the answer choices are linearly independent, so at most one of the answer choices can be correct. The eigenvectors of A lie in the null space of $A-\lambda I$ for $\lambda = 1+i$. We compute

$$A-\lambda I = \begin{pmatrix}-1-i & -2\\ 1 & 1-i\end{pmatrix} \overset{RREF}{\Longrightarrow} \begin{pmatrix}1 & 1-i\\ 0 & 0\end{pmatrix} \implies x_1 = (-1+i)x_2$$

so the λ-eigenspace is $\operatorname{span}(\begin{pmatrix} -1+i \\ 1 \end{pmatrix})$. Multiplying this vector by $1+i$ gives us

$$(1+i)\begin{pmatrix} -1+i \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 1+i \end{pmatrix}$$

which is still an eigenvector for λ.

Question 13. Recall that the area of $T(S)$ is $|\det(T)| * \operatorname{area}(S)$. The determinant of T's matrix A is

$$\det(A) = 5 \cdot 4 - 7 \cdot 6 = -22$$

so the area is $|-22| \cdot 1 = 22$.

Question 14. The characteristic polynomials of the matrices are as follows.

1. $$\det(\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix}) = \lambda^2 + 1$$

2. $$\det(\begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix}) = \lambda^2 - 1$$

3. $$\det(\begin{pmatrix} -1-\lambda & 0 \\ 0 & 1-\lambda \end{pmatrix}) = (-1-\lambda)(1-\lambda) - 0(0) = \lambda^2 - 1$$

4. $$\det(\begin{pmatrix} -1-\lambda & 0 \\ 0 & -1-\lambda \end{pmatrix}) = (-1-\lambda)(-1-\lambda) - 0(0) = \lambda^2 + 2\lambda + 1$$

Question 15

NO

If $1+i$ and $-1+i$ are eigenvalues, then so are their conjugates $1-i$ and $-1-i$.

Together with the eigenvalue 0, we now have five eigenvalues for a 4×4 matrix, which is impossible.

Question 16

Answer : $\begin{pmatrix} 24 \\ 40 \end{pmatrix}$

Assuming $A^n \begin{pmatrix} 34 \\ 30 \end{pmatrix}$ converges,

$A^n \begin{pmatrix} 34 \\ 30 \end{pmatrix}$ approaches $64 \begin{pmatrix} 12/32 \\ 20/32 \end{pmatrix} = \begin{pmatrix} 24 \\ 40 \end{pmatrix}$, as

n goes to infinity.

Here, 64 is the sum $34+30$.

Question 17

(a) NO

A 2×2 matrix A is diagonalizable
if and only if
the sum of the geometric multiplicities of the eigenvalues of A is equal to 2.
(In our case, we have 1).

(b) MAYBE

Take, for example

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ which is diagonalizable, and}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ which is NOT diagonalizable.}$$

(c) YES.

Since C is a triangular matrix, its eigenvalues are the diagonal entries of C.
Since the three eigenvalues are distinct, we can find three linearly independent eigenvectors. So, C is diagonalizable.

Question 18

Answer: $\lambda_1 = 4, \quad \lambda_2 = -4, \quad \lambda_3 = 0.$

λ_1 is the eigenvalue corresponding to the eigenvector $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$, that is the first column of C.

This means that $A\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \lambda_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$

Computing

$$\begin{pmatrix} 0 & 0 & -4 \\ 2 & 2 & 2 \\ -6 & -6 & -2 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 4 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix},$$ we see

that $\lambda_1 = 4.$

Similarly,

$A\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} = -4\begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix}$ and $A\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = 0\cdot\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ give

$\lambda_2 = -4$ and $\lambda_3 = 0.$

Question 19

Answer: $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}$

Finding a basis for the -2 – eigenspace of of the matrix $A = \begin{pmatrix} -1 & 3 & -1 \\ 0 & -2 & 1 \\ 1 & 3 & 0 \end{pmatrix}$ means finding a basis for $\mathrm{Nul}\,(A + 2I_3)$.

We solve $(A + 2I_3)x = 0$:

$$A + 2I_3 = \begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 1 \\ 1 & 3 & 2 \end{pmatrix} ,$$

its RREF is $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Parametric form of the solutions is $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$.

$Nul(A + 2I_3) = span \left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}$.

So, $\left\{ \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for -2-eigenspace.

<u>Question 20</u>

(a) FALSE.

A 4×4 matrix has at most four eigenvalues.

(b) TRUE

We know that 0 is not an eigenvalue of A.

(c) TRUE

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Question 21

Answer:

$$A\begin{pmatrix}5\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

$$A\begin{pmatrix}5\\-1\end{pmatrix}=\begin{pmatrix}-5\\1\end{pmatrix},$$

$$A\begin{pmatrix}5\\-1\end{pmatrix}=\begin{pmatrix}\sqrt{50}\\-\sqrt{2}\end{pmatrix}.$$

By definition $\begin{pmatrix}5\\-1\end{pmatrix}$ is an eigenvector of A if

$A\begin{pmatrix}5\\-1\end{pmatrix}=\lambda\begin{pmatrix}5\\-1\end{pmatrix}$ for some scalar λ.

In cases above, we use $\lambda = 0, -1, \sqrt{2}$, respectively.