Math 1553: Fall 2020

Practice Midterm 1 Solutions

September 16, 2020

Problem 2.

Here, our matrix is already in reduced row-echelon form, which is the first step to writing a solution in parametric vector form. The next step is to write out the equations: $x_1 - 3x_2 = 0$ and $x_3 = 5$. Then, we solve for each pivot variable, which in this problem are x_1 and x_3, in terms of constants and our free variables, which here is x_2. This gives us $x_1 = 3x_2$ and $x_3 = 5$. We then substitute these into the vector form of a solution and break this expression up into a sum of a constant vector and vectors times each free variable:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 + 3x_2 \\ 0 + 1x_2 \\ 5 + 0x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

This is parametric vector form, but the solution to this problem oddly asks for just two variables in each vector rather than three. Therefore, we infer that they are asking for only the expressions for our pivot variables, as we have two of these, x_1 and x_3, and getting expressions for pivot variables is one major goal of parametric vector form. After all, free variables will just stay as themselves. Taking out the middle row, we have that

$$\begin{pmatrix} x_1 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

so our answers are $A = 0$, $B = 5$, $C = 3$, and $D = 0$.

Problem 3.

In this problem, we are asked for the form of the solution set to an equation of the form $Ax = b$, when we are given the form of the solution set $Ax = 0$. We know that if $Ax = b$ is consistent, then the solution set to $Ax = b$ is a translate of the solution set to $Ax = 0$. Therefore, our solution set here will either be a line or no solution, and as A has 3 columns, our solution set lives in $\mathbb{R}^3$. Unfortunately, all possible answer choices are a line in $\mathbb{R}^3$, so we cannot yet figure out the answer.

Now, we recall that if a_1, a_2, and a_3 are the columns of A, then $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = a_1x_1 + a_2x_2 + a_3x_3$. In the problem, they give us that $A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, so the first column of A, a_1, must be exactly $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$. Similarly, the fact that $A\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ tells us that $a_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$. So our matrix $A = \begin{pmatrix} 1 & 2 & a_{13} \\ -1 & 0 & a_{23} \\ 2 & 1 & a_{33} \end{pmatrix}$ for some numbers a_{13}, a_{23}, a_{33}. Luckily, we can finish this problem even without knowing those three values, as the vectors we need to multiply A by have a 0 in their third row. We can now use process of elimination to see whether any of the four points in the solutions to the problem is a solution to $Ax = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$. Using standard matrix multiplication on the matrix for A that we have determined, we see that $A\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$, $A\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $A\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$, and $A\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$. Therefore, $Ax = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ has at least one solution, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, so it is consistent, and the solution set must then be a line in $\mathbb{R}^3$ including $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (but not including $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, or $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$).

Problem 4.

A set of vectors are linearly dependent if and only if when you put those vectors as the rows of a matrix and bring that matrix to row-echelon form, the resulting matrix has a zero row. You make the vectors the rows rather than the columns because the process of row reduction is basically the same as finding linear dependencies between the rows. Therefore, our task now is to row reduce the matrix $\begin{pmatrix} 1 & 2 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & h \end{pmatrix}$ (our vectors have now become the rows). We start by replacing row 2 with row 2 plus row 1 and row 3 with row 3 minus twice row 1 in order to zero out the first column. This gives us the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & -4 & h-2 \end{pmatrix}$. Then, we replace row 3 with row 3 plus row 2 in order to zero out the second column, giving us the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 1 \\ 0 & 0 & h-1 \end{pmatrix}$. This matrix has a zero row if and only if $h = 1$, so $h = 1$ would cause these vectors to be linearly dependent.

Alternatively, one could have noted that if the third vector were in the span of the first two, then certainly the three vectors would then be linearly dependent. And since $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, setting $h = 1$ would then make the third vector to be a linear combination of the first two (1 times vector 1 plus -1 times vector 2).

Problem 5.

If two vectors span a plane, then those two vectors are linearly independent, as a plane is two dimensional. However, even if a set of three vectors are pairwise independent (that is, any two of them are linearly independent) all three of them together may be dependent. This can be true if all three vectors lie on the same plane, but no two are collinear. For instance, choosing my favorite plane, the xy-plane, we could take the vectors $\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\begin{pmatrix}1\\1\\0\end{pmatrix}$, and $\begin{pmatrix}0\\1\\0\end{pmatrix}$. Any two of these can be combined to make any vector on the xy-plane, so any two span a plane. However, we see that $\begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, a linear dependency relation that tells us that these three vectors together are not independent, so the answer is no. In fact, here we know that any vector lies in the span of the other two, the xy-plane.

Problem 6.

This problem asks us to compute a matrix product. While imagining our high school teacher telling us to move our left hand across while we move our right hand down, we find that

$$\begin{pmatrix}1 & 2 & 0\\0 & -2 & 3\\-1 & 0 & 2\end{pmatrix}\begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}1\cdot 1 + 2\cdot 2 + 0\cdot -1\\0\cdot 1 + -2\cdot 2 + 3\cdot -1\\-1\cdot 1 + 0\cdot 2 + 2\cdot -1\end{pmatrix} = \begin{pmatrix}1+4+0\\0+-4+-3\\-1+0+-2\end{pmatrix} = \begin{pmatrix}5\\-7\\-3\end{pmatrix}$$

What a nice break from all that abstract stuff!

Problem 7.

One can notice the variables x, y, and z in our two equations, so in order to write these linear equations as a matrix it is first helpful to add 0 times any variable that is not being used into each equation. This gives the equations $x + -2y + 0z = 4$ and $0x + 3y + -5z = -1$. We then stack these two equations on top of each other, make our coefficients into a matrix, separate out our variables, and put our solutions on the other side of the equal sign to get

$$\begin{pmatrix}1 & -2 & 0\\0 & 3 & -5\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}4\\-1\end{pmatrix}$$

It is helpful to check one's work by multiplying out these matrices to see that this does indeed give our original two equations back. For problems like this, it is often helpful to do this for all answer choices. Here, we note that two of the other answer choices do not even have legal dimensions for matrix equations, as a $m \times n$ matrix always needs to be multiplied by a $n \times 1$ vector to result in a $m \times 1$ vector, while the other answer choice gives equations that use only x and y but not z.

Problem 8.

A matrix equation is consistent if and only if when we row reduce the augmented matrix, there is no pivot in the augmenting column. So here, we row reduce

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 3 & -1 & 3 \\ 2 & 3 & -5 & 5 \end{array}\right) \xrightarrow{R_3=R_3-2R_1} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 3 & -1 & 3 \\ 0 & 3 & -1 & 3 \end{array}\right) \xrightarrow{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -2 & 1 \\ 0 & 3 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Now, we see that our matrix is in row-echelon form, and we have no pivot in the augmented column, meaning our equation is consistent. In fact, we can see that our third row was just twice the first row plus the second, so it was redundant, while the first and second rows can both be satisfied, with x_3 as a free variable and x_1 and x_2 respectively dependent on them.

Problem 9.

First, note that A has six columns, so the solution set will live inside $\mathbb{R}^6$, and thus will have dimension no more than 6. But can the solution set have exactly dimension 6? Equivalently, can we set A such that every vector in $\mathbb{R}^6$ be a valid solution to $Ax = 0$? (as $\mathbb{R}^6$ itself is the only space inside $\mathbb{R}^6$ with dimension 6.) The answer is yes. If we set $A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, then for any vector $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}$, $Ax = 0$ is satisfied, and thus our solution set is $\mathbb{R}^6$, which has dimension 6. Therefore, it is false that the dimension of the solution set is at most 5. As an aside, if the problem changed to $Ax = b$ for some $b \neq 0$, then in fact the statement in the problem would become true, as there is no matrix that maps all vectors in $\mathbb{R}^6$ to a given non-zero b vector.

Problem 10.

This question asks us to select all that apply, so we need to consider each answer individually.

The first statement says that in a linearly dependent set of vectors, every vector is a linear combination of the others. This is false, as a linear dependency could only be among a subset of the total dependent set of vectors. For instance, the vectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ are linearly dependent, as $2\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + -1\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, but $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$. Notably, there is some linear dependence relation between these vectors, and in any linear dependence relation between these vectors, the coefficient of

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ will be 0. This is required to make our example work, as if there was a dependency with a non-zero coefficient on our last vector, we could then solve for that vector as a linear combination of the others. So this statement is false.

For the second statement, let m be the number of rows in A and let n be the number of columns of A. The equation $Ax = b$ is consistent if and only if b is in the span of the columns of A. So if $Ax = b$ is consistent for every b, then the n columns of A span the space they live in, which is $\mathbb{R}^m$ (as each column has m coordinates). Then if the columns are also linearly independent, they form a linearly independent set of n vectors that spans $\mathbb{R}^m$. The only way that that can happen is if $n = m$, so A must be square and the second statement is true. You can think about this with $m = 3$: for vectors to span $\mathbb{R}^3$, there must be at least three of them, and for vectors to be linearly independent in $\mathbb{R}^3$, there can't be more than three of them. So if both are true, there is exactly three of them.

The third statement is true, as the zero vector is in fact in the span of every set of vectors. The span of vectors $v_1, \cdots, v_n$ is anything you can write as $c_1v_1 + \cdots + c_nv_n$ for some real numbers $c_1, \cdots, c_n$. Just set $c_1 = \cdots = c_n = 0$ and you can form the 0 vector, so the zero vector is in the span. Therefore, this statement is true

The final statement is false, according to the "overview" section of the textbook. If that doesn't convince you, any future engineering, business, or computer science classes you take certainly will. Check out how many classes have linear algebra as a direct or indirect prereq. Hope you've been paying attention!

Problem 11.

We assume that no cars stop in intersections, and thus for each intersection, the number of cars entering each intersection must equal the number of cars leaving the intersection. Therefore, the left intersection gives us that $2 + 3 = y + z$, the bottom intersection gives us that $9 + y = x + 1$, and the top intersection gives us that $x + z = 6 + 7$. Moving all variables to the left side of these equations and all constants to the right, we get that $-y - z = -5$, $-x + y = -8$, and $x + z = 13$. Adding in the coefficients of 0, our system of linear equations is $0x + -1y + -1z = -5$, $-1x + 1y + 0z = -8$, and $1x + 0y + 1z = 13$. As an augmented matrix, this gives us $\left(\begin{array}{ccc|c} 0 & -1 & -1 & -5 \\ -1 & 1 & 0 & -8 \\ 1 & 0 & 1 & 13 \end{array}\right)$ which we then row reduce:

$$\left(\begin{array}{ccc|c} 0 & -1 & -1 & -5 \\ -1 & 1 & 0 & -8 \\ 1 & 0 & 1 & 13 \end{array}\right) \xrightarrow{swap(R_1,R_3)} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ -1 & 1 & 0 & -8 \\ 0 & -1 & -1 & -5 \end{array}\right) \xrightarrow{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & -1 & -1 & -5 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 13 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Then we see that our reduced matrix has pivots in the x and y columns only, meaning that our system is consistent (no pivot in the augmented column) and has one free variable, z (as z is the only non-solution column with no pivot).

It makes sense that z must be a free variable here, as if road z were shut down, all cars travelling on it could instead travel along road y and then road x. So we have one degree of freedom, which allows us to choose how many of the cars that much travel from the left intersection to the top one go along z, and how many go on y and then x. And it makes sense that our system is consistent, as if you count the total number of cars entering and leaving the edges of the diagram, they are equal.

Problem 12.

For four vectors in $\mathbb{R}^4$ to span $\mathbb{R}^4$, they must be linearly independent. For two vectors to span a plane (2 dimensional), they must be linearly independent. So this problem is asking if we can find four linearly independent vectors, any two of which are also linearly independent. Well we don't have to search too hard for those: just choose any four linearly independent vectors in $\mathbb{R}^4$! To keep it simple, lets go with

$$\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix}\}$$

Problem 13.

To check which answer choices are true, it is helpful to first row reduce A. All of the answer choices involve the solution set of $Ax = b$, so as row reducing does not change the solution set, it will not change our answer here.

$$\begin{pmatrix}1&2&4\\2&0&1\\3&1&0\end{pmatrix}\xrightarrow{R_2=R_2-2R_1,R_3=R_3-3R_1}\begin{pmatrix}1&2&4\\0&-4&-7\\0&-5&-12\end{pmatrix}$$
$$\xrightarrow{R_3=4R_3-5R_2}\begin{pmatrix}1&2&4\\0&-4&-7\\0&0&4\cdot -12+-5\cdot -7\end{pmatrix}=\begin{pmatrix}1&2&4\\0&-4&-7\\0&0&-13\end{pmatrix}$$

Now, we can note that A already has a pivot in every row, so for any b, $A|b$ will not have a pivot in the augmented column and will thus be consistent. Additionally, as every column has a pivot, we have no free variables, so every $A|b$ will have a unique solution for every b, including 0. Therefore, it is false that $Ax = 0$ has multiple solutions, it is false that there is a b such that $Ax = b$ has no solution, it is true that $Ax = b$ is always consistent, and it is false that A has fewer than three pivots when row reduced (it has exactly three).

Q. 14

We have

$$\underset{A}{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 5 \end{pmatrix}}, \quad \underset{B}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 5 & 6 & 9 \end{pmatrix}}, \quad \underset{C}{\begin{pmatrix} 1 & 0 & 0 \\ 18 & 2 & 0 \\ 20 & 1 & 12 \end{pmatrix}}$$

A: Since column vectors are in $\mathbb{R}^2$, they cannot span $\mathbb{R}^3$.

B: RREF

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 5 \\ 3 & 5 & 6 & 9 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 3 & 3 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since there are only two linearly independent column vectors, the columns cannot span $\mathbb{R}^3$.

C: RREF

$$\begin{pmatrix} 1 & 0 & 0 \\ 18 & 2 & 0 \\ 20 & 1 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 12 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus columns of C span $\mathbb{R}^3$. //

Q.15 Ones in RREF are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$ //

Q.16

The point satisfies all of the equations in the system. //

Q.17 True

$\mathbb{R}^5$ can have at most 5 linearly independent vectors. //

Q.18

Problem 18**.**

Two vectors are linearly dependent if and only if they are multiples of each other/collinear. Here, we see that the second vector is $-\pi$ times the first, so they are collinear and thus dependent. In fact they both lie on the line $y = -x$. One linearly dependency relation between them is $\pi \begin{pmatrix} 1 \\ -1 \end{pmatrix} + 1 \begin{pmatrix} -\pi \\ \pi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Problem 19**.**

Our matrix has four columns, and so our solution set lives in $\mathbb{R}^4$. If a system is consistent, the dimensionality of our solution set is the number of columns without a pivot, which here is 4 total columns - 2 pivot columns = 2 non-pivot columns. And our system here must be consistent, as the homogenous equation $Ax = 0$ always at least has the zero vector as a solution. Therefore, our solution set is a two-dimensional space – a plane – inside $\mathbb{R}^4$.

Problem 20**.**

a) We see that this is in reduced row-echelon form with pivots in the first and third column. Therefore, only x_2 is a free variable, as the second column is the only column (not including the solution column) without a pivot.

b) This is in reduced row-echelon form with a pivot in every column (not including the augmenting column). Therefore, we have no free variables.

Problem 21**.**

If we have a system of two equations in four variables, our matrix will have four columns (not counting the solution column) and two rows. Therefore, after row reduction we can have at most two pivots. This means that at least two columns must have no pivot and therefore must be free variables. Since we must have at least two free variables, options 1 and 3 (with 1 and 0 free variable respectively) are not possible. We can give equations that result in the second, fourth, and fifth possible solution sets:

Second solution set: $1x_1 + 0x_2 + 0x_3 + 0x_4 = -1$ and $0x_1 + 1x_2 + 0x_3 + 3x_4 = 9$

Fourth solution set: $x_1 + 9x_2 + -3x_3 + 2x_4 = 8$ and $-1x_1 + -9x_2 + 3x_3 + -2x_4 = -8$

Fifth solution set: $x_1 + x_2 + x_3 + x_4 = 1$ and $x_1 + x_2 + x_3 + x_4 = 2$ (the sum cannot equal both 1 and 2)