Math 1553 Quiz 10 Solution

Question 2

Which of the following are correct diagonalizations of the matrix $\begin{bmatrix} 2 & 6 \\ 0 & -1 \end{bmatrix}$?

Ans: $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}^{-1}$ and

$\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$ and

$\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & -1 \end{bmatrix}^{-1}$

Because the matrix is triangular, we know that the eigenvalues are $\lambda = 2, -1$. The matrix D can therefore look like $\begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$ Solving for the eigenvector associated with $\lambda = 2$, we get A - 2I = $\begin{bmatrix} 0 & 6 & |0 \\ 0 & -3 & |0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & |0 \\ 0 & 0 & |0 \end{bmatrix} \rightarrow x_1 = x_1, x_2 = 0$. A basis for the 2-eigenspace is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or any multiple of this vector (such as $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$). Solving for the eigenvector associated with $\lambda = -1$, we get A - (-1)I = $\begin{bmatrix} 3 & 6 & |0 \\ 0 & 0 & |0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & |0 \\ 0 & 0 & |0 \end{bmatrix} \rightarrow x_1 = -2x_2, x_2 = x_2$. A basis for the -1-eigenspace is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, or any multiple of this vector (such as $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$). The eigenvectors in matrix C must appear in the same order (column-wise) as the respective eigenvalues in the matrix D.

Question 3

Suppose that A is a 5x5 matrix with charateristic polynomial $(1-\lambda)^2(3-\lambda)^2(\pi-\lambda)$ and also that A is diagonalizable. What is the dimension of the 1-eigenspace of A?

Ans: 2

Because the matrix A is diagonalizable, the algebraic multiplicity (number of times λ appears as a root in the characteristic polynomial) must match the geometric multiplicity (dimension of the eigenspace associated with λ) for each value of λ. 1 appears as a root of the characteristic polynomial twice, therefore the geometric multiplicity of λ = 1 / dimension of the 1-eigenspace is 2.

Question 4

Suppose A is a 2x2 matrix whose entries are real numbers, and suppose A has eigenvalue 1-i with corresponding eigenvector $\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$. Which of the following must be true?

Ans: A has eigenvalue 1+i with eigenvector $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$

From the slides, "if λ is an eigenvalue with eigenvector v, then $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{v}$." In this case, $\overline{\lambda} = \overline{1-i} = 1+i$ and $\overline{v} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$

Question 5

If A is a diagonalizable 10 x 10 matrix, then A must have 10 distinct eigenvalues.

Ans: False

An nxn matrix could be diagonalizable without n distinct eigenvalues if at least one of the eigenvalues has a multiplicity greater than one. For instance, the the 10 x 10 identity matrix I_{10} is diagonalizable but only has one distinct eigenvalue, λ = 1.

Question 6

Suppose that A is a 4 x 4 matrix with eigenvalues 0, 1, and 2, where eigenvalue 2 has geometric multiplicity 2 (meaning that the dimension of the 2-eigenspace is 2). Which of the following statements much be true?

Ans: A is diagonalizable, A is not invertible

Because the geometric multiplicity of λ = 2 is 2, and eigenvalues 0, 1 must have at least a 1-dimensional eigenspace, we have the 4 linearly independent eigenvectors needed to complete the diagonalization of a 4x4 matrix.
Because 0 is an eigenvalue of A, the equation Ax=0 does not have only the trivial solution. Therefore, A is not invertible.