Problem 2. Solve for the unknown a in the vector equation

$$\begin{pmatrix} 2 \\ a \end{pmatrix} + \begin{pmatrix} -1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Your answer should be a number.

Solution. Recalling that vector addition is defined component-wise, the given equation can be rewritten as

$$\begin{pmatrix} 1 \\ a+7 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since two vectors are equal if and only if their corresponding components are equal, it follows that $a + 7 = 1$, i.e. $a = \boxed{-6}$. $\square$

Problem 3. Consider the following two vectors v and w:

$$v = \begin{pmatrix} 6 \\ -6 \\ 0 \end{pmatrix}, \quad w = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}.$$

For which value of h is the following vector in the span of v and w?

$$\begin{pmatrix} 2 \\ h \\ -4 \end{pmatrix}$$

Your answer should be a number.

Solution. We can reformulate the problem as finding the value of h such that the augmented system

$$\left(\begin{array}{cc|c} 6 & 0 & 2 \\ -6 & 2 & h \\ 0 & -1 & -4 \end{array}\right)$$

has at least one solution. Note that the three given vectors go into the columns of the augmented matrix.

Performing the row operations $R_2 = R_2 + R_1$ and then $R_3 = R_3 + \frac{1}{2}R_2$ brings us to row echelon form:

$$\left(\begin{array}{cc|c} 6 & 0 & 2 \\ 0 & 2 & h+2 \\ 0 & 0 & -4 + \frac{1}{2}(h+2) \end{array}\right).$$

Examining the last row (because to the left of the vertical bar it consists of all zeros), we see that the system has at least one solution if and only if $-4 + \frac{1}{2}(h+2) = 0$, i.e. $h = \boxed{6}$. $\square$

Problem 4. Fill in the blank in the following matrix product.

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ \square \end{pmatrix}$$

Your answer should be a number.

Solution. Using the "row-column rule" for matrix-vector multiplication (see section 2.3 in the textbook for a refresher), we get

$$\square = \begin{pmatrix} -1 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = -1 \cdot 1 + 0 \cdot 2 + 1 \cdot 3 + 0 \cdot 4 = \boxed{2}.$$

$\square$

Problem 5. Suppose that A is a matrix, and that we can row reduce it to the following matrix:

$$\begin{pmatrix} 1 & 2 & -3 \\ 0 & -7 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Which of the following statements are necessarily true about A? Select all that apply.

1. The span of the columns of A is $\mathbb{R}^3$
2. The matrix equation $Ax = b$ is consistent for every b in $\mathbb{R}^3$.
3. There is a b in $\mathbb{R}^3$ so that $Ax = b$ is inconsistent.
4. The span of the columns of A is a line in $\mathbb{R}^3$
5. The matrix equation $Ax = 0$ is consistent

Solution. Because the given row echelon form of A does not have any rows consisting of all zeros, we see that the augmented system $(A \,|\, b)$ is consistent for every b in $\mathbb{R}^3$. In other words, the matrix equation $Ax = b$ is consistent for every b in $\mathbb{R}^3$, i.e. statement $\boxed{2}$ is true. In yet different words, the span of the columns of A is $\mathbb{R}^3$, i.e. statement $\boxed{1}$ is true. This immediately implies that statements 3 and 4 are false since each of them directly contradicts either statements 1 or 2. As for statement 5, note that $x = 0$ is always a solution to the matrix equation $Ax = 0$, so statement $\boxed{5}$ is true. $\square$

Problem 6. Suppose that A is a 3×2 matrix. Which of the following statements must be true about A?

1. There is vector b so that $Ax = b$ is inconsistent
2. The span of the columns of A is a plane in $\mathbb{R}^3$
3. The span of the columns of A is $\mathbb{R}^3$

Solution. Because A has 2 columns and each column can contain at most one pivot position, we know that A has at most 2 pivot positions. But A has 3 rows, so it follows that there is at least one row containing no pivot positions.

Now recall the following theorem from section 2.3 in the textbook:

Theorem 1. *Let A be an $m \times n$ (non-augmented) matrix. The following are equivalent:*

1. *$Ax = b$ has a solution for all b in $\mathbb{R}^m$.*
2. *The span of the columns of A is all of $\mathbb{R}^m$.*
3. *A has a pivot position in every row.*

We conclude from this theorem and our observation that A has at least one row containing no pivot positions that the $\boxed{\text{first statement}}$ in the problem is true and the third statement is false. To see that the second statement is false as well, consider the counterexample

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This works as a counterexample because the span of the columns of A is just $\{(0, 0, 0)\}$, which in particular is not a plane. $\square$